Geometry of Relativistic Spacetime Physics

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Abstract

This article introduces and describes the mathematical structures and frameworks needed to understand the modern fundamental theory of Relativistic Spacetime Physics. The self-referential and self-contained nature of Mathematics provides enough power to prescribe a rigorous language needed to formulate the building components of the standard Einstein's General Theory of Relativity like Spacetime, Matter, and Gravity, along with their behaviors and interactions. In these notes, we will introduce and understand these abstract components, starting with defining the arena of smooth manifolds and then adding the necessary and sufficient differential geometric structures needed to build the primers to the General Theory of Relativity.

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1 Topological spaces

Sets defined via the Zermelo–Fraenkel set theory with the axiom of choice, need to have enough structure in them in order to access a notion of continuity for which we need a minimal construct called a topology.

Definition 1.1 (Topology)**.** Let M be a set. A *topology* $\mathcal{O}$ is a subset of the power set of M, i.e., $\mathcal{P}(M)$ satisfying the axioms

1. $\emptyset, M \in \mathcal{O}$,
2. If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$,
3. If A is an arbitrary index set (possibly uncountable), and $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{O}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$.

For any set M, the topology $\mathcal{O}_{\text{chaotic}} \coloneqq \{\emptyset, M\}$ is called the *chaotic* topology of M, and the topology $\mathcal{O}_{\text{discrete}} \coloneqq \mathcal{P}(M)$ is called the *discrete* topology of M.

Example 1. For $M = \mathbb{R}^d$ for some $d \in \mathbb{N}$, consider $\mathcal{O}_{\text{standard}} \subseteq \mathcal{P}(\mathbb{R}^d)$ defined via the steps

1. Define the soft-ball $B_r(p)$ for $r \in \mathbb{R}_+$ and $p \in \mathbb{R}^d$ as $B_r(p) \coloneqq \left\{ (q_i)_{i=1}^d \;\middle|\; \sum_{i=1}^d (q_i - p_i)^2 \le r^2 \right\}$. Note that the definition of the soft-ball need not require the set to be a vector space.
2. $U \in \mathcal{O}_{\text{standard}} :\iff \forall\, p \in U \;\exists\, r \in \mathbb{R}_+ \ni B_r(r) \subseteq U.$

This topology is the usual topology on $\mathbb{R}^d$ unless mentioned explicitly. The standard topology is only defined for Euclidean spaces as we usually happen to work with this topology in Euclidean geometry.

The pair $(M, \mathcal{O})$ is called a *topological space*. If $U \in \mathcal{O}$, we call U an *open set*. The notion of openness is defined through the topology $\mathcal{O}$. We call a set $A \subset M$ a *closed set* if $M \setminus A \in \mathcal{O}$. It should be noted that openness and closedness do not imply each other.

1.1 Continuous maps

A topology buys us a notion of continuous maps. Continuity of a map depends on the topologies chosen on the domain set and the target set.

Definition 1.2 (Continuous maps)**.** Let the domain set M be equipped with a topology $\mathcal{O}_M$, and the target set N be equipped with a topology $\mathcal{O}_N$. Then a map $f : M \to N$ is called a *continuous map* (w.r.t. the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$) if

$$\forall\, V \in \mathcal{O}_N,\ \operatorname{preim}_f(V) \in \mathcal{O}_M,$$

where for any $V \subseteq N$,

$$\operatorname{preim}_f(V) \coloneqq \{m \in M \mid f(m) \in V\}.$$

Continuous maps preserve the structure of topologies. Note that the definition of preim_f is the same as f^{-1} is f is a bijective map. A map therefore is continuous if and only if the preimages of (all) open sets in the target space are open sets in the domain space.

Example 2. For $M = \{1, 2\}$, $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $N = \{1, 2\}$, $\mathcal{O}_N = \{\emptyset, \{1, 2\}\}$, the function $f : M \to N$ defined as $f(1) = 2$ and $f(2) = 1$ is a continuous map w.r.t. the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$. The map f^{-1} is not continuous.

Definition 1.3 (Composition of continuous maps)**.** Let $f : M \to N$ and $g : N \to P$, then $g \circ f : M \to P$ is defined as $m \mapsto (g \circ f)(m) \coloneqq g(f(m))$.

Theorem 1.4. *If f and g are continuous maps between M and N, and N and P, with respect to the topologies $\mathcal{O}_M, \mathcal{O}_N$, and $\mathcal{O}_P$ of sets M, N, and P, then $g \circ f$ is also continuous with respect to the topologies $\mathcal{O}_M$ and $\mathcal{O}_P$.*

Proof. Observe that for any $V \in \mathcal{O}_P$, we have

$$\operatorname{preim}_{g \circ f}(V) = \operatorname{preim}_f\big(\operatorname{preim}_g(V)\big)$$

Because g is a continuous map, we have $\operatorname{preim}_g(V) \in \mathcal{O}_N$. Now since f is also a continuous map, we have $\operatorname{preim}_f\big(\operatorname{preim}_g(V)\big) \in \mathcal{O}_M$. □

1.2 Inheriting a topology

Topologies can be inherited from given topological space(s).

Definition 1.5 (Subset topology)**.** For a topological space $(M, \mathcal{O}_M)$, and for a subset $S \subseteq M$, we can define $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$, as

$$\mathcal{O}\big|_S \coloneqq \{U \cap S \mid U \in \mathcal{O}_M\}$$

called the *subset topology.* $\mathcal{O}\big|_S$ satisfies Definition 1.1, and therefore is a topology in itself.

Constructing a subset topology allows us to argue that the restriction of continuous functions defined on a super-set is also a continuous function.

2 Topological Manifolds

There are numerous notions of topological spaces for classification. For space-time physics, we focus on topological spaces $(M, \mathcal{O})$ that can be *charted*, analogously to how the surface of the Earth is charted in an *atlas*.

Definition 2.1 (Topological manifolds)**.** A topological manifold $(M, \mathcal{O})$ is called a d-dimensional topological manifold if

$$\forall\, p \in M\colon\ \exists\, U \in \mathcal{O}\colon p \in U \subseteq M,\ \exists\, x\colon U \mapsto x(U) \subseteq \mathbb{R}^d,$$

such that

1. x is invertible, i.e., $x^{-1}\colon x(U) \mapsto U$, and
2. x and x^{-1} are continuous with respect to the topologies $\mathcal{O}$ and $\mathcal{O}_{\text{standard}}$.

Example 1. For a torus $M \coloneqq \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^3$, we equip the manifold with the topology $\mathcal{O} \coloneqq \mathcal{O}_{\text{standard}}\big|_M$. Then $(M, \mathcal{O})$ is a 2-dimensional topological manifold.

Example 2. The set M defined as the shape of the letter '$\mathcal{Y}$' is a subset of $\mathbb{R}^2$ and can be equipped with a topology $\mathcal{O} \coloneqq \mathcal{O}_{\text{standard}}\big|_M$, but it fails to be a topological manifold. Note that there exists a point in M where it is not possible to find an invertible and continuous (in both directions) map on the open neighborhood $U \cap M \in \mathbb{R}^2$ into either $\mathbb{R}$ or $\mathbb{R}^2$. Therefore $(M, \mathcal{O})$ is neither a 1-dimensional or a 2-dimensional topological manifold.

Definition 2.2 (Chart)**.** For a topological manifold $(M, \mathcal{O})$, the pair containing an open set U and its corresponding map x satisfying Definition 2.1, is called a *chart*.

Definition 2.3. For a d-dimensional topological manifold $(M, \mathcal{O})$,

1. a set $\mathcal{A} \coloneqq \{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ for some index set A is called an *atlas* if $M = \bigcup_{\alpha \in A} U_\alpha$.
2. For for $p \in U$, $x(p) = (x^1(p), x^2(p), \ldots, x^d(p))$, the components $\left\{x^i\colon U \to \mathbb{R}\right\}_{i=1}^{d}$ are called the *coordinate maps* of the chart (U, x), and
3. $x^i(p) \in \mathbb{R}$ is called the i-th coordinate of the point p with respect to the chosen chart (U, x).

Definition 2.4 (Chart transition map)**.** Consider two charts (U, x) and (V, y) of a d-dimensional topological manifold $(M, \mathcal{O})$ with overlapping regions, i.e., $U \cap V \neq \emptyset$. Consider a point $p \in U \cap V$, then the continuous map $y \circ x^{-1}\colon x(U \cap V) \to y(U \cap V)$ is called the *chart transition map*.

Refer to the commutative diagram in Figure 1.

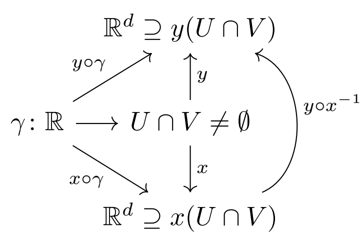


Figure 1: Chart transition maps

All the charts can be *glued* together via the chart transition maps which are simply some maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. The chart transition maps can therefore provide all the global topological information about the manifold.

It is often desirable to define properties e.g., continuity, of real world objects e.g., a curve $\gamma\colon \mathbb{R} \to M$ by judging suitable conditions not on the real world object itself through the topologies, but on the chart-representative of that real world object, i.e., of $x \circ \gamma$ for a chart (U, x) of a topological manifold $(M, \mathcal{O})$.
To see that the imaginative introduction of charts to study the real world is consistent for continuity, it is sufficient to observe the continuity of chart transition maps and that the function composition operation is associative.
The property of differentiability of a curve $\gamma\colon \mathbb{R} \to U$ cannot be directly defined from just the topology of the manifold (as it only provides continuity) due to the absence of addition and multiplication. Since the chart transition maps need not be differentiable, we do not have the similar consistency for differentiability across charts because composing a continuous map on a differentiable map could only preserve continuity but not differentiability. We will resolve this through adding more structure to the atlas of the topological manifold which we will discuss in § 4.

3 Multilinear Algebra

Definition 3.1 ($\mathbb{R}$-Vector space)**.** A $\mathbb{R}$-vector space $(V, +, \cdot)$ is

1. a set V,
2. Addition: $+ \colon V \times V \to V$, and
3. S-multiplication: $\cdot \colon \mathbb{R} \times V \to V$,

which satisfies for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$,

1. Commutative law for $+$: $v + w = w + v$,
2. Associative law for $+$: $(u + v) + w = u + (v + w)$,
3. Neutral element for $+$: $\exists\, 0 \in V \colon \forall\, v \in V : v + 0 = v$
4. Inverse element for $+$: $\forall\, v \in V\ \exists\, (-v) \in V : v + (-v) = 0$
5. Associative law for $\cdot$: $\lambda \cdot (\mu \cdot v) = (\lambda \cdot_{\mathbb{R}} \mu) \cdot v$,
6. First Distributive law: $(\lambda +_{\mathbb{R}} \mu) \cdot v = \lambda \cdot v + \mathbb{R}\mu \cdot v$,
7. Second Distributive law: $\lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
8. Unitary law: $1 \cdot v = v$,

where $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$ are $\mathbb{R}$-addition and $\mathbb{R}$-multiplication respectively.

From hereon, we will write $\mathbb{R}$-vector spaces as just vector spaces.

Definition 3.2 (Vector)**.** An element of the vector space $(V, +, \cdot)$ is referred to as a *vector*. Note that an element of V alone is not a vector without it having a vector space structure over it.

Example 1. For $N \in \mathbb{Z}_+$, the set $P := \left\{ p \colon (-1, +1) \to \mathbb{R} \,\middle|\, p(x) = \sum_{n=0}^{N} p_n \cdot x^n, p_n \in \mathbb{R}\ \forall\ n \in [N] \cup \{0\} \right\}$, with the addition $+$ defined as

$$\begin{aligned} +_P \colon P \times P &\to P \\ (p, q) &\mapsto p + q \\ \text{where, } (p +_P q)(x) \colon p(x) +_{\mathbb{R}} q(x) &\quad \forall\ x \in (-1, +1), \end{aligned}$$

and the scalar multiplication $\cdot$ defined as,

$$\begin{aligned} \cdot_P \colon \mathbb{R} \times P &\to P \\ (\lambda, q) &\mapsto \lambda \cdot_P p \\ \text{where, } (\lambda \cdot_P p)(x) = \lambda \cdot_{\mathbb{R}} p(x) &\quad \forall\ x \in (-1, +1), \end{aligned}$$

is a vector space.

Just like we studied maps between topological spaces which respect the topological structure between their topologies, we have a similar notion of maps between vector spaces that respect the vector space structure between two vector spaces.

Definition 3.3 (Linear maps)**.** If $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ are two vector spaces, then as map

$$\varphi \colon V \to W$$

is called *linear* if

1. $\varphi(v +_V \tilde{v}) = \varphi(v) +_W \varphi(\tilde{v})$, and
2. $\varphi(\lambda \cdot_V v) = \lambda \cdot_W \varphi(v)$.

Then, we define the linear (and multilinear) map φ as $\varphi\colon V \xrightarrow{\sim} W$.

Example 2. Consider the map on $(P, +_P, \cdot_P)$ defined as

$$\begin{aligned} \delta\colon P &\to P \\ p &\mapsto \delta(p) = p', \end{aligned}$$

where p' is the derivative of the polynomial $p \in P$. The map δ is linear because we have for $p, q \in P$,

$$\begin{aligned} \delta(p +_P q) &= (p +_P q)' = p' +_P q' = \delta(p) +_P \delta(q), \\ \text{and, } (\lambda \cdot_P p)' &= \lambda \cdot_P p' = \lambda \cdot_P \delta(p). \end{aligned}$$

Theorem 3.4. *If we have linear maps between vector spaces $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$ and $(U, +_U, \cdot_U)$ as*

$$\varphi\colon V \xrightarrow{\sim} W, \qquad \text{and} \qquad W \xrightarrow{\sim} U \tag{3.1}$$

then $\varphi \circ \psi\colon V \xrightarrow{\sim} W$.

Example 3. $\delta \circ \delta\colon P \xrightarrow{\sim} P$.

From hereon, we will avoid writing subscripts on the addition and scalar multiplication operation of vector spaces and infer them from the context and usage.

3.1 Vector space of Homomorphisms

Consider the vector spaces $(V, +, \cdot)$ and $(W, +\cdot)$, then

$$\mathrm{Hom}(V, W) \coloneqq \left\{ \varphi\colon V \xrightarrow{\sim} W \right\},$$

with the addition operation defined as

$$\begin{aligned} \oplus\colon \mathrm{Hom}(V, W) \times \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W), \\ (\varphi, \psi) &\mapsto \varphi \oplus \psi, \\ \text{where, } (\varphi \oplus \psi)(v) &\coloneqq \varphi(v) + \varphi(v), \end{aligned}$$

and the scalar multiplication defined as

$$\begin{aligned} \odot\colon \mathbb{R} \times \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W), \\ (\lambda, \varphi) &\mapsto \lambda \odot \varphi, \\ \text{where, } (\lambda \odot \varphi)(v) &\coloneqq \lambda \cdot \varphi(v), \end{aligned}$$

then $(\mathrm{Hom}(V, W), \otimes, \odot)$ is a vector space.

Example 4. $(\mathrm{Hom}(P, P), \otimes, \odot)$ is a vector space, and $\delta \in \mathrm{Hom}(P, P)$, and $\underbrace{\delta \circ \ldots \circ \delta}_{M \in \mathbb{Z}_+ \text{times}} \in \mathrm{Hom}(P, P)$. Then, we also have the mixed differential operator elements of the form $5 \odot \delta \oplus (\delta \circ \delta) \in \mathrm{Hom}(P, P)$.

3.2 Dual vector space

Consider a vector space $(V, +, \odot)$, then we define the set

$$V^* \coloneqq \left\{ \varphi\colon V \xrightarrow{\sim} \mathbb{R} \right\} = \mathrm{Hom}(V, \mathbb{R}) \tag{3.2}$$

with the addition and scalar multiplication as defined in § 3.1, $(V^*, \otimes, \odot)$ is called the *dual vector space* to V, and an element $\varphi \in V^*$ is called a *covector* under this setup.

Example 5. Consider the linear map $I\colon P \xrightarrow{\sim} \mathbb{R}$, i.e., $I \in P^*$ defined as

$$I(p) := \int_0^1 \mathrm{d}x \; p(x),$$

is a linear map since

$$\begin{aligned} I(p+q) &= \int_0^1 \mathrm{d}x \; (p+q)(x) = I(p) + I(q), \\ \text{and } I(\lambda \cdot p) &= \lambda \cdot I(p). \end{aligned}$$

Therefore $I = \int_0^1 \mathrm{d}x$ is a covector.

3.3 Tensors

Definition 3.5 (Tensors)**.** Let $(V, +, \cdot)$ be a vector space. Then an (r, s)-tensor T over V is a multi-linear map

$$T\colon \underbrace{V^* \times \ldots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \ldots \times V}_{s \text{ copies}} \xrightarrow{\sim} \mathbb{R}.$$

Example 6. If T is a $(1, 1)$ tensor, then from multilinearity, it satisfies for $\lambda \in \mathbb{R}, \varphi, \psi \in V^*, v, w \in V$,

$$\begin{aligned} T(\varphi + \psi, v) &= T(\varphi, v) + T(\psi, v), & T(\varphi, v + w) &= T(\varphi, v) + T(\varphi, w), \\ T(\lambda \cdot \varphi, v) &= \lambda \cdot T(\varphi, v), \quad \text{and} & T(\varphi, \lambda \cdot v) &= \lambda \cdot T(\varphi, v). \end{aligned}$$

A $(1, 1)$-tensor $T\colon V^* \times V \xrightarrow{\sim} \mathbb{R}$ can be thought of to provide us a map on V to itself. Consider the function ϕ_T defined as

$$\begin{aligned} \phi_T\colon V &\xrightarrow{\sim} (V^*)^* \\ v &\mapsto T(\cdot, v)\colon V^* \xrightarrow{\sim} \mathbb{R} \end{aligned}$$

If the dimension of the vector space is finite, then $(V^*)^* = V$. Similarly, given a linear map $\phi\colon V \xrightarrow{\sim} V$, we can define a $(1, 1)$-tensor

$$\begin{aligned} T_\phi\colon V^* \times V &\xrightarrow{\sim} \mathbb{R} \\ (\varphi, v) &\mapsto \varphi(\phi(v)). \end{aligned}$$

Then we have that given a tensor T, we have $T = T_{\phi_T}$, and given a linear map ϕ, we have $\phi = \phi_{T_\phi}$.

Example 7. The map

$$\begin{aligned} g\colon P \times P &\xrightarrow{\sim} \mathbb{R} \\ (p, q) &\mapsto \int_{-1}^1 \mathrm{d}x \; p(x) q(x), \end{aligned}$$

which is the inner product of the two polynomial inputs, is a $(0, 2)$-tensor.

Theorem 3.6. *If $(V, +, \cdot)$ is a vector space, then $\varphi \in V^* \iff \varphi\colon V \xrightarrow{\sim} \mathbb{R} \iff \varphi$ is a $(0, 1)$-tensor.*

Theorem 3.7. *If $(V, +, \cdot)$ is a vector space, then $v \in V = (V^*)^* \iff v\colon V^* \xrightarrow{\sim} \mathbb{R} \iff v$ is a $(1, 0)$-tensors.*

3.4 Bases

Definition 3.8 (Hamel-Bases)**.** Let $(V, +, \cdot)$ be a vector space, then $B \subseteq V$ is called a *Hamel basis* if

$$\forall\ v \in V\ \exists!\ F = \{f_1, \dots, f_n\} \subseteq B:\ \exists!\ \left\{v^1, v^2, \dots, v^n\right\} \subset \mathbb{R} : v = v^1 f_1 + \dots + v^n f_n.$$

The basis B in the above definition can have infinitely many elements, but given a vector we should be able to write it as a linear combination of finitely many elements from that basis. Otherwise, we have a series and need a notion of convergence to even define it. To define the notion of convergence we at least need a topology on the vector space, which leads us the definition of so called *Schauder basis*.

Definition 3.9 (Dimension of a vector space)**.** If there exists a basis B of a vector space $(V, +, \cdot)$ with finitely many elements, say $d \in \mathbb{N}$, then we call $\dim V := d$.

Remark 3.10. Let $(V, +, \cdot)$ be a finite dimensional vector space. Having chosen a basis $\{e_1, \dots e_n\}$ of $(V, +, \cdot)$, we may uniquely associate $v \mapsto (v^i \in \mathbb{R})_{i=1}^d$ called the *components* of v with respect to the chosen basis, where $v^1 e_1 + \dots v^n e_n = v$.

3.5 Basis for the dual space

We can choose a basis $\{e_1, \dots, e_n\}$ for a vector space $(V, +, \cdot)$, and we can choose a basis $\left\{\epsilon^1, \dots, \epsilon^n\right\}$ for V^* independent of the basis of V. However, it is more economical to require that once a basis $e_1, \dots, e_n$ on V has chosen, that the basis on V^*, $\left\{\epsilon^1, \dots, \epsilon^n\right\}$ satisfies

$$\epsilon^a(e_b) = \delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

This uniquely defines a choice of $\left\{\epsilon^1, \dots, \epsilon^n\right\}$ for $(V^*, +, \cdot)$ from the choice of $\{e_1, \dots, e_n\}$ for $(V, +, \cdot)$. If a basis $\left\{\epsilon^1, \dots, \epsilon^n\right\}$ of $(V^*, +, \cdot)$ satisfies this, it is called the *dual basis* of the dual space V^*.

Example 8. For $N = 3$, the vector space $(P, +, \cdot$ with basis $\{e_0, e_1, e_2, e_3\}$ is a basis if $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$ and $e_3(x) = x^3$. The dual basis $\left\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\right\}$ defined as $\epsilon^a := \frac{1}{a!}\partial^a \big|_{x=0}$, satisfies $\epsilon^a(e_b) = \delta^a_b$ for $a, b \in [3] \cup \{0\}$.

3.6 Components of tensors

Definition 3.11. Let T be a (r, s)-tensor on a finite dimensional vector space $(V, +, \cdot)$, and let $\{e_1, \dots, e_n\}$ be the basis of $(V, +, \cdot)$, and $\left\{\epsilon^1, \dots, \epsilon^n\right\}$ be the basis of $(V^*, +, \cdot)$ satisfying $\epsilon^a(e_b) = \delta^a_b\ \forall\ a, b \in [d]$, then define the $(r + s)^{\dim V}$ many real numbers as

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_s} := T(\epsilon^{i_1}, \epsilon^{i_2}, \dots, \epsilon^{i_r}, e_{j_1}, e_{j_2}, \dots, e_{j_s}) \in \mathbb{R} \quad \forall\ i_1, \dots, i_r, j_1, \dots, j_s \in [\dim V].$$

These numbers are called the components of the tensor with respect to the chosen basis.

Knowing the components and the basis, one can reconstruct the entire tensor.

Example 9. If T is a (1,1)-tensor, then $T^i_j = T(\epsilon^i, e_j)$ for $i, j \in [\dim V]$, and for any $\varphi \in V^*$ and $v \in V$ with components $(\varphi_i)_{i=1}^{\dim V}$ and $(v^j)_{i=1}^{\dim V}$ respectively, we have

$$\begin{aligned} T(\varphi, v) &= T\left(\sum_{i=1}^{\dim V} \varphi_i \epsilon^i, \sum_{j=1}^{\dim V} v^j e_j\right) \\ &= \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim V} T^i_j \varphi_i v^j \\ &=: T^i_j \varphi_i v^j, \end{aligned}$$

where the last equality is by using the Einstein's summation convention where multiple repeated indices are summed over appropriately.

4 Differentiable Manifolds

We studied topological manifolds in § 2, where it allows us to define continuity of curves. Above continuity, we also want to associate it with a a notion of velocity. The structure of a topological manifold is not enough to provide us differentiability of curves. We wish to define a notion of differentiable objects (e.g., curves, functions, maps across manifolds).
Consider a curve $\gamma\colon \mathbb{R} \to M$ on a d-dimensional topological manifold $(M, \mathcal{O})$. We can consider each chart (U, x) and talk about the curve $x \circ \gamma$ on $x(U) \subseteq \mathbb{R}^d$ on that portion of the manifold. We can then then lift the notion of differentiability of a curve on $\mathbb{R}^d$ to a notion of differentiability of a curve on M. We are left to see if this notion is well-defined under the change of charts.

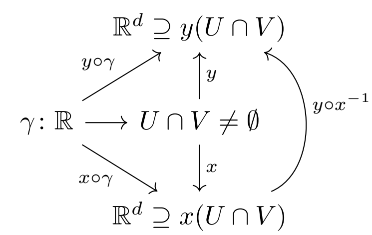


Figure 2: Differentiability of chart transition maps

To see this, consider two charts (U, x) and (V, y) with a non-empty intersection, then the map $y\circ\gamma = y\circ(x^{-1}\circ x)\circ\gamma = \left(y \circ x^{-1}\right) \circ (x \circ \gamma)$. We can ensure that $x \circ \gamma\colon \mathbb{R} \to \mathbb{R}^d$ is $\mathbb{R}$-differentiable, but we can only ensure the $\mathbb{R}$-continuity of the function $y \circ x^{-1}$. Therefore, their composition $y \circ \gamma$ may only be continuous but not differentiable. Refer to the commutative diagram in Figure 2. There is a similar problem when we consider functions on topological manifolds or map across topological manifolds. Therefore this strategy does not work out. As a remedy, from the atlas of the topological manifold which is maximal, we can choose only the subset of charts under which all the transition functions are differentiable. Such a restricted atlas restricts us to use only a certain kind of charts and that turns out to be fine.

Definition 4.1 (Compatible charts)**.** For a d-dimensional topological manifold, two charts (U, x) and (V, y) are called ❀-*compatible* if either

1. $U \cap V = \emptyset$
2. $U \cap V \neq \emptyset$, then the chart transition map $y \circ x^{-1}\colon x(U \cap V) \to y(U \cap V)$ and $x \circ y^{-1}\colon y(U \cap V) \to x(U \cap V)$ have the ❀ property from $\mathbb{R}^d$ to $\mathbb{R}^d$.

If the ❀ property is differentiability property, then we call the charts to be differentially compatible.

Definition 4.2 (Compatible atlas)**.** An Atlas $\mathcal{A}_{❀}$ is a ❀ compatible atlas if any two charts in $\mathcal{A}_{❀}$ are ❀ compatible.

For any ❀ property on $\mathbb{R}^d$, a ❀ manifold is a triple $(M, \mathcal{O}, \mathcal{A}_{❀})$, where $\mathcal{A}_{❀}$ is a subset of the maximal atlas $\mathcal{A}_{\text{maximal}}$, and $(M, \mathcal{O})$ is a d-dimensional topological manifold.

❀	Property
C^0	$C^0\left(\mathbb{R}^d \to \mathbb{R}^d\right)$: continuous maps
C^1	$C^1\left(\mathbb{R}^d \to \mathbb{R}^d\right)$: differentiable (exactly once) maps
C^k	$C^k\left(\mathbb{R}^d \to \mathbb{R}^d\right)$: $k \in \mathbb{Z}_+$ times continuously differentiable
D^k	$C^k\left(\mathbb{R}^d \to \mathbb{R}^d\right)$: $k \in \mathbb{Z}_+$ times differentiable
C^∞	$C^\infty\left(\mathbb{R}^d \to \mathbb{R}^d\right)$: $k \in \mathbb{Z}_+$ times continuously differentiable for all $k \in \mathbb{Z}_+$
C^ω	there exists a multi-dimensional Taylor expansion. Note $C^\omega \subsetneq C^\infty$
$\mathbb{C}^\infty$	Charts with maps that pairwise satisfy Cauchy-Riemann equations, for even dimensional manifolds

Table 1: Examples of ❀ properties

Continuous differentiability is easy to check just from the repeated continuous differentiability in various direction, i.e., from the existence and continuity of repeated partial derivatives. Differentiability does not guarantee the continuity of partial derivatives.

Theorem 4.3 (Whitney's theorem)**.** *Any C^k atlas $\mathcal{A}_{C^k}$ for $k \in \mathbb{N}$ of a topological manifold contains a C^∞ atlas.*

Thus, without loss of any generality always consider C^∞ manifold called *smooth m manifold* unless we wish to define Taylor expandability or complex differentiability etc.

Definition 4.4 (Smooth manifold)**.** $(M, \mathcal{O}, \mathcal{A})$ is a smooth manifold if $(M, \mathcal{O})$ is a topological manifold and $\mathcal{A}$ is a C^∞ atlas.

4.1 Diffeomorphisms

Definition 4.5 (Isomorphism)**.** A set M and a set N are called to be *isomorphic* (set-theoretically), if there exists a bijection $\phi\colon M \to N$ between them. We denote this as $M \cong N$.

Example 1.

1. $\mathbb{N} \cong \mathbb{Z}$
2. $\mathbb{N} \cong \mathbb{Q}$
3. $\mathbb{N} \ncong \mathbb{R}$

Definition 4.6 (Homeomorphic)**.** Two topologies $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ are topologically isomorphic, or homeomorphic, if there exists a bijection $\phi\colon M \to N$ between the sets and ϕ and ϕ^{-1} are continuous with respect to their topologies. This is denoted as $(M, \mathcal{O}_M) \cong (N, \mathcal{O}_N)$.

Definition 4.7 (Isomorphism between vector spaces)**.** Two vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ are vector space isomorphic if there exists a bijection $\phi\colon V \to W$ such that ϕ and ϕ^{-1} are linear.

Definition 4.8 (Diffeomorphism)**.** Two C^∞ manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ of dimension d and e respectively are said to be *diffeomorphic* if there exists a bijection $\phi\colon V \to W$ such that ϕ and ϕ^{-1} are both C^∞ maps. To check if they are C^∞ maps, consider the charts (U, x) and (V, y) of the two respective manifolds, then we need the map $y \circ \phi \circ x^{-1}\colon x(U) \to y(V)$ and $x \circ \phi^{-1} \circ y^{-1}\colon y(V) \to x(U)$ to be in C^∞.

Refer to the commutative diagram in Figure 3.

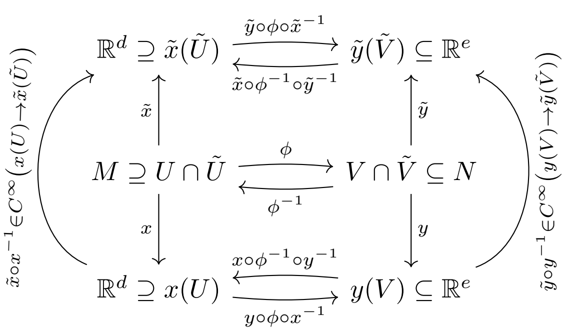


Figure 3: C^∞ compatibility for diffeomorphism

Theorem 4.9. *The number of C^∞ manifold one can make out of a C^0 manifold $(M, \mathcal{O}, \mathcal{A}_{C^0})$ (if any) up to diffeomorphism are as in Table 2.*

dim M	Number of C^∞ manifolds up to diffeomorphisms from a given topological manifold	
1	1	Moise Radon theorems
2	1	
3	1	
4	uncountably infinitely many	
5	finitely many	Surgery theory
6	finitely many	
$\vdots$	finitely many	

Table 2: Number of C^∞ manifolds up to diffeomorphisms from a given topological manifold

5 Tangent Spaces

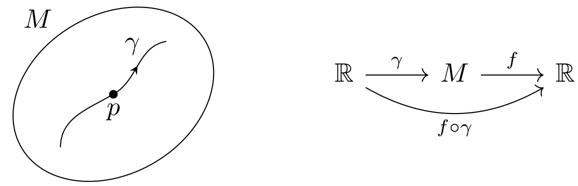


Figure 4: Velocity of a curve

Definition 5.1 (Velocities)**.** We consider a smooth manifold $(M, \mathcal{O}, \mathcal{A})$, and a curve $\gamma\colon \mathbb{R} \to M$ that is at least C^1. Suppose $\gamma(\lambda_0) = p$ for some $\lambda_0 \in \mathbb{R}$. The velocity of γ at p is a linear map $v_{\gamma,p}\colon C^\infty(M) \xrightarrow{\sim} \mathbb{R}$, where $(C^\infty(M), \oplus, \odot)$ is a vector space on the set of all smooth functions on the manifold equipped with addition operation $\oplus$ defined as $(f \oplus g)(p) = f(p) + g(p)$, and $(\lambda \odot g)(p) = \lambda \cdot g(p)$ for any two elements $f, g \in C^\infty(M)$, where $v_{\gamma,p}$ is defined as

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0). \tag{5.1}$$

5.1 Tangent vector space

Definition 5.2. For each point $p \in M$, we define the set T_pM called the *tangent space* to M at the point p which is defined as

$$T_pM := \{v_{\gamma,p}\colon \text{smooth } \gamma\colon \mathbb{R} \to M\}$$

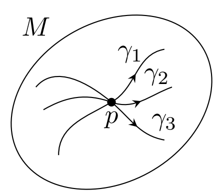


Figure 5: The tangent space as the set of velocities at a point

Remark 5.3. There are analogous visualizations that are preferred where a tangent vector space is imagined lying on top of the manifold touching it at the point, and both are considered to be embedded in an even larger dimensional manifold. But that is not how we should think of it technically as when we will consider the whole universe as a manifold, an object outside of it doesn't make a strict real sense. To make this distinction explicit and clear in diagrams, we will use colored straight arrows with arrow heads at the end to denote tangent vectors (and vector fields), and black dashed lines to denote objects lying in the tangent space at the point of origin of the arrows and dashed lines.

T_pM can be made into a vector space. We define the addition $\oplus$ is defined as

$$\begin{aligned} \oplus\colon T_pM \times T_pM &\to \mathrm{Hom}(C^\infty(M), \mathbb{R}) \\ (v_{\gamma,p} \oplus v_{\delta,p})(f) &:= v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall\, f \in C^\infty(M) \end{aligned}$$

and the scalar multiplication $\odot$ as

$$\begin{aligned} \odot\colon \mathbb{R} \times T_pM &\to \mathrm{Hom}(C^\infty(M), \mathbb{R}) \\ (\alpha \odot v_{\gamma,p})(f) &:= \alpha \cdot v_{\gamma,p}(f). \end{aligned}$$

The reason why both the operations map to $\mathrm{Hom}(C^\infty(M), \mathbb{R})$ generally is that in order to have the addition lie in the same vector space, the result (which indeed lies in $\mathrm{Hom}(C^\infty(M), \mathbb{R})$) needs to be generated by some curve. It remains to be shown that

1. $\exists\, \sigma\colon \mathbb{R} \to M : v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$, and

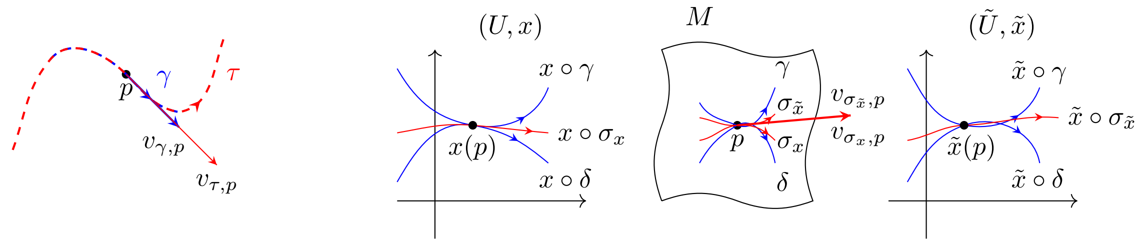


Figure 6: Construction of $v_{\tau,p}$, $v_{\sigma_x,p}$ and $v_{\sigma_{\tilde{x}},p}$

2. $\exists\, \tau\colon \mathbb{R} \to M : \alpha \odot v_{\gamma,p} = v_{\tau,p}$.

We can construct the curve $\tau\colon \mathbb{R} \to M$ as

$$\begin{aligned} \tau\colon \mathbb{R} &\to M \\ \lambda &\mapsto \tau(\lambda) \coloneqq \gamma(\alpha \cdot \lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \end{aligned}$$

where $\mu_\alpha\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} \mu_\alpha\colon \mathbb{R} &\to \mathbb{R} \\ r &\mapsto \alpha \cdot r + \lambda_0. \end{aligned}$$

To check, observe

$$\begin{aligned} \tau(0) &= \gamma(\lambda_0) \\ v_{\tau,p}(f) &= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) = (f \circ \gamma)'(\mu_\alpha(0)) \cdot \mu_\alpha'(0) = \alpha \cdot (f \circ \gamma)'(\lambda_0) = \alpha \cdot v_{\gamma,p}. \end{aligned}$$

We make a choice of a chart (U, x), and construct $\sigma_x\colon \mathbb{R} \to M$ as

$$\begin{aligned} &\sigma_x\colon \mathbb{R} \to M \\ &\sigma_x(\lambda) \coloneqq x^{-1}((x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0)) \end{aligned}$$

where $\gamma(\lambda_0) = \delta(\lambda_1) = p$. To check, observe

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_1) - (x \circ \gamma)(\lambda_0)) = \delta(\lambda_1) = p, \\ v_{\sigma_x,p}(f) &= (f \circ \sigma_x)'(0) \\ &= \big((f \circ x^{-1}) \circ (x \circ \sigma_x)\big)'(0) \\ &= \Big((x \circ \sigma_x)^i\Big)' \cdot \big(\partial_i\big(f \circ x^{-1}\big)\big)((x \circ \sigma_x)(0)) \\ &= \bigg(\Big((x \circ \gamma)^i\Big)'(\lambda_0) + \Big((x \circ \delta)^i\Big)'(\lambda_1)\bigg) \cdot \big(\partial_i\big(f \circ x^{-1}\big)\big)(x(p)) \\ &= \Big((x \circ \gamma)^i\Big)'(\lambda_0) \cdot \big(\partial_i\big(f \circ x^{-1}\big)\big)(x(p)) + \Big((x \circ \delta)^i\Big)'(\lambda_1) \cdot \big(\partial_i\big(f \circ x^{-1}\big)\big)(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_1) \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f) \quad f \in C^\infty(M) \\ \implies v_{\sigma_x,p} &= v_{\gamma,p} \oplus v_{\delta,p}, \end{aligned}$$

independent of the chart chosen. Thus $(T_pM, \oplus, \odot)$ is indeed a vector space.

5.2 Components of a vector with respect to a chart

Let $(U, x) \in \mathcal{A}_{\text{smooth}}$. Let $\gamma\colon \mathbb{R} \to U$, and $\gamma(0) = p \in M$. Then for $f \in C^\infty(M)$,

$$\begin{aligned} v_{\gamma,p} f &= (f \circ \gamma)'(0) = \big((f \circ x^{-1}) \circ (x \circ \gamma)\big)(0) \\ &= \big((x \circ \gamma)^i\big)'(0) \cdot \partial_i(f \circ x^{-1})(x(p)) \end{aligned}$$

As notation, we will use the notation to denote $\partial_i(f\circ x^{-1})(x(p))$ as $\left(\frac{\partial f}{\partial x^i}\right)_p$, which does not make formal sense since $f\colon M\to\mathbb{R}$ and $x^i\colon M\to\mathbb{R}$, but does follow all the rules of the partial derivative operator. We will also denote $\big((x\circ\gamma)^i\big)'$ as $\dot{\gamma}_x^i$, therefore allowing us to write

$$\begin{aligned} v_{\gamma,p}f &= \dot{\gamma}_x^i(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p f \\ \implies v_{\gamma,p} &= \dot{\gamma}_x^i(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p, \end{aligned}$$

where $\dot{\gamma}_x^i(0)$ denote the components of the velocity with respect to the basis elements $\left(\frac{\partial}{\partial x^i}\right)_p$ of the space T_pM, with respect to which the components need to be understood. This basis is a chart induced basis of T_pM. We could have chosen a different chart and it would give us a different basis of the tangent space.

5.3 Chart induced basis

If $(U,x)\in\mathcal{A}_{\text{smooth}}$ is a chart of a d-dimensional smooth manifold, then $\left(\frac{\partial}{\partial x^1}\right)_p,\dots,\left(\frac{\partial}{\partial x^d}\right)_p\in T_pU\subseteq T_pM$, and constitute a basis of T_pU. It remains to be show that they are linearly independent, i.e.,

$$\lambda^i\left(\frac{\partial}{\partial x^i}\right)_p \overset{!}{=} 0 \qquad \implies \lambda^i = 0\ \forall\ i\in[d].$$

This is easy to see since if we apply this to the differentiable maps $x^j : U\to\mathbb{R}$ for $j\in[d]$, we get

$$\begin{aligned} 0 \overset{!}{=} \lambda^i\left(\frac{\partial}{\partial x^i}\right)_p x^j &= \lambda^i\,\partial_i\big(x^j\circ x^{-1}\big)(x(p)) \\ &= \lambda^i\delta_i^j = \lambda^j. \end{aligned}$$

Therefore the set $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}_{i=1}^d$ is a basis of T_pM.

Corollary 5.3.1. $\dim T_pM = d = \dim M$, *where the first dimension is the vector space dimension, and the second is the topological manifold dimension.*

As terminology, when we say $X\in T_pM$, then it is equivalent to saying that $\exists\ \gamma\colon\mathbb{R}\to M$ such that $X = v_{\gamma,p}$. Note that there can be multiple curve that give rise to the same abstract tangent vector. $X\in T_pM$ also means that $\exists\ X^1,\dots,X^d\in\mathbb{R} : X = X^i\left(\frac{\partial}{\partial x^i}\right)_p$, where the components $\{X^i\}_{i=1}^d$ are with respect to the chosen basis which can come from a chart.

5.4 Change of vector components under change of chart

The tangent vector does not change under the change of chart, but the components do. Let (U,x) and (V,y) be overlapping charts, and take a point $p\in U\cap V$. Let $X\in T_pM$, then

$$X = X_{(x)}^i\left(\frac{\partial}{\partial x^i}\right)_p, \quad \text{and} \quad X = X_{(y)}^j\left(\frac{\partial}{\partial y^j}\right)_p. \tag{5.2}$$

To study the change of components, consider

$$\begin{aligned} \left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i\big(f\circ x^{-1}\big)(x(p)) \\ &= \partial_i\big(f\circ y^{-1}\circ y\circ x^{-1}\big)(x(p)) \\ &= \Big(\partial_i\big(y\circ x^{-1}\big)^j(x(p))\Big)\cdot\partial_j\big(f\circ y^{-1}\big)(y(p)) \\ &= \partial_i\big(y^j\circ x^{-1}\big)(x(p))\cdot\partial_j\big(f\circ y^{-1}\big)(y(p)) \end{aligned}$$

$$= \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p f$$

$$\implies \left(\frac{\partial}{\partial x^i}\right)_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p. \tag{5.3}$$

Therefore combining Equation (5.2) and Equation (5.3) we get

$$X^i_{(x)} \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p = X^j_{(y)} \left(\frac{\partial}{\partial y^j}\right)_p,$$

$$\implies \left(X^i_{(x)} \left(\frac{\partial y^j}{\partial x^i}\right)_p - X^j_{(y)}\right) \left(\frac{\partial}{\partial y^j}\right)_p = 0$$

$$\implies X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)} \quad \forall\, j \in [d]. \tag{5.4}$$

For any to pair of charts, the matrix $\left[\left(\frac{\partial y^j}{\partial x^i}\right)_p\right]_{j,i=1}^d$ evaluated at the point p is a constant matrix that determines the change in vector components at that point as a constant linear map. The global chart transformation can be non-linear.

5.5 Cotangent spaces

Consider the dual vector space of T_pM, $T_p^*M \coloneqq \left\{\varphi \colon T_pM \xrightarrow{\sim} \mathbb{R}\right\}$.

Example 1. For $f \in C^\infty(M)$, then we can define

$$(\mathrm{d}f)_p \colon T_pM \xrightarrow{\sim} \mathbb{R},$$
$$X \mapsto (\mathrm{d}f)_p(X) \coloneqq Xf.$$

Then $(\mathrm{d}f)_p \in T_p^*M$ is called the *gradient* of f at $p \in M$. We can calculate the components of the gradient as a $(0,1)$ tensor or a *covector* with respect to a chart induced basis for the chart (U, x).

$$((\mathrm{d}f)_p)_j \coloneqq (\mathrm{d}f)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right)$$
$$= \left(\frac{\partial f}{\partial x^j}\right)_p = \partial_j\left(f \circ x^{-1}\right)(x(p)).$$

Theorem 5.4. *Consider a chart (U, x) i.e., $\left\{x^i \colon U \to \mathbb{R}\right\}_{i=1}^d$ of a d-dimensional smooth manifold $(M, \mathcal{O}, \mathcal{A})$, then $\left\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \dots (\mathrm{d}x^d)_p\right\}$ is a basis of T_p^*M. In fact, it is the dual basis of the dual space T_p^*M, i.e., $(\mathrm{d}x^a)\left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial x^a}{\partial x^b}\right)_p = \delta^a_b$.*

5.6 Change of components of a covector under change of chart

If $\omega \in T_pM^*$, then for any two intersecting charts (U, x) and (U, y) such that $p \in U \cap V$, we have

$$\omega = \omega_{(x)_i} (\mathrm{d}x^i)_p, \quad \text{and} \quad \omega = \omega_{(y)_j} (\mathrm{d}y^j)_p,$$

then using chain rule just like as we did in § 5.4, we get

$$\omega_{(y)_j} = \left(\frac{\partial x^i}{\partial y^j}\right)_p \omega_{(x)_i}.$$

6 Fields

In § 5 we dealt with tangent space at a fixed point. In general we would like to talk about tangent vectors at every point of the manifold. To introduce this, we need the theory of bundles.

6.1 Bundles

Definition 6.1 (Bundle)**.** A *bundle* is a triple (E, π, M),

$$E \xrightarrow{\pi} M,$$

where E is a smooth manifold called the total space. M is a smooth manifold which is called the base space. π is a surjective smooth map between E and M called the projection map.

Example 1. For $E = \mathbb{R} \times \mathbb{S}^1$, $M = \mathbb{S}^1$, and $\pi\colon E \to M$ defined as in Figure 7.

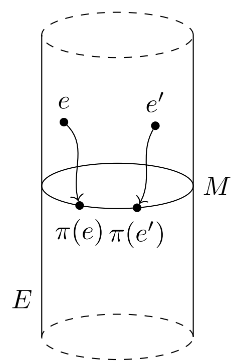


Figure 7: The bundle $(\mathbb{R} \times \mathbb{S}^1, \pi, \mathbb{S}^1)$.

Definition 6.2 (Fibre)**.** Let (E, π, M) be a bundle, and $p \in M$, then the fibre over p is $\mathrm{preim}_\pi(\{p\})$.

Definition 6.3 (Section)**.** A *section* $\sigma\colon M \to E$ of a bundle (E, π, M) is a map which satisfies $\pi \circ \sigma = \mathrm{id}_M$.

A section is a *field*.

6.2 Tangent bundle of a smooth manifold

Let $(M, \mathcal{O}, \mathcal{A})$ be a d-dimensional smooth manifold, then

1. define as a set, tangent bundle of the manifold is $TM \coloneqq \dot{\bigcup}_{p\in M} T_pM$,
2. define the surjective map π defined as

$$\begin{aligned} \pi\colon TM &\to M \\ X &\mapsto p, \end{aligned}$$

   such that $p \in M$ is the unique point such that $X \in T_pM$,

3. construct the coarsest topology on TM such that the map π is (just) continuous (i.e., the initial topology with respect to π), as $\mathcal{O}_{TM} \coloneqq \{\mathrm{preim}_\pi(U) \mid U \in \mathcal{O}\}$. It can be shown that $\mathcal{O}_{TM}$ is indeed a topology, and
4. construct a C^∞ atlas on TM from the C^∞ atlas of M, as $\mathcal{A}_{TM}\colon \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$, where for a chart $(U, x) \in \mathcal{A}$, $TU \coloneqq \dot{\cup}_{p\in U} T_pM$,

$$\begin{aligned} \xi_x\colon TU &\to \mathbb{R}^{2d} \\ X &\mapsto \big((x^1 \circ \pi)(X), \ldots, (x^d \circ \pi)(X), (\mathrm{d}x^1)_{\pi(X)}(X), \ldots, (\mathrm{d}x^d)_{\pi(X)}(X)\big), \end{aligned}$$

such that $X \in T_pU$, and its inverse ξ_x^{-1} is

$$\begin{aligned} \xi_x^{-1} \colon \xi_x(TU) &\to TU \\ (\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) &\mapsto \beta^i \left(\frac{\partial}{\partial x^i}\right)_{\underbrace{x^{-1}(\alpha^1, \dots, \alpha^d)}_{\pi(X)}}. \end{aligned}$$

To check if the atlas $\mathcal{A}_{TM}$ is C^∞ compatible, consider any two charts (TU, ξ_x) and (TV, ξ_y) in $\mathcal{A}_{TM}$, such that $U \cap V \neq \emptyset$, and check

$$\begin{aligned} &\big(\xi_y \circ \xi_x^{-1}\big)\big(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d\big) \\ =&\xi_y \left(\beta^i \left(\frac{\partial}{\partial x^i}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}\right) \\ =&\left(\dots, (y^i \circ \pi)\left(\beta^m \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}\right), \dots, (\mathrm{d}y^i)_{\pi\left(\beta^m \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}\right)} \left(\beta^m \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}\right), \dots\right) \\ =&\left(\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \beta^m (\mathrm{d}y^i)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \left(\left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}\right), \dots\right) \\ =&\left(\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \beta^m \left(\frac{\partial y^i}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}, \dots\right) \end{aligned}$$

is a smooth map since $(y^i \circ x^{-1})$ is a smooth map, and $\left(\frac{\partial y^i}{\partial x^m}\right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} = \partial_m\big(y^i \circ x^{-1}\big)\big(x\big(x^{-1}(\alpha^1, \dots, \alpha^d)\big)\big) = \partial_m\big(y^i \circ x^{-1}\big)\big(\alpha^1, \dots, \alpha^d\big)$ is a smooth map.

Therefore, $(TM, \mathcal{O}_{TM}, \mathcal{A}_{TM})$ is a smooth manifold and therefore (TM, π, M) is a bundle, called the *tangent bundle*.

6.3 Vector fields

Definition 6.4 (Smooth vector field)**.** A *smooth vector field* $\chi \colon M \to TM$ is a smooth map, that is a section of a tangent bundle (TM, π, M). Then we have $\pi \circ \chi = \mathrm{id}$.

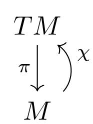


Figure 8: Smooth vector field as a section of (TM, π, M)

A vector field is therefore an smooth association of a vector at each point of a manifold.

6.4 The $C^\infty(M)$-module $\Gamma(TM)$

C^∞ is the set of all smooth functions on M. We know that it is a vector space since we can add two smooth functions, or multiply a smooth function with a real scalar to get a smooth function. C^∞ is a ring since a smooth function need not have a smooth multiplicative inverse. We define the set $\Gamma(TM)$ as

$$\Gamma(TM) \coloneqq \{\chi \colon M \to TM \mid \chi \text{ is a smooth section}\}.$$

This set can be equipped with an addition $\oplus$ defined as

$$\begin{aligned} \oplus \colon \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (\chi, \tilde{\chi}) &\mapsto \chi \oplus \tilde{\chi} \\ \text{such that} \quad (\chi \oplus \tilde{\chi})(f) &\coloneqq \chi f +_{C^\infty(M)} \tilde{\chi} f \end{aligned}$$

for all $f \in C^\infty(M)$, where $\chi f := \chi(p)f$ for any $\chi \in \Gamma(TM)$, and a $C^\infty(M)$ multiplication $\odot$ defined as

$$\begin{aligned} \odot\colon C^\infty(M) \times \Gamma(TM) &\to \Gamma(TM) \\ (g, \chi) &\mapsto g \odot \chi \\ \text{such that} \quad (g \cdot \chi)f &:= g \cdot_{C^\infty(M)} \chi f \end{aligned}$$

for all $f \in C^\infty(M)$. Therefore $(\Gamma(TM), \oplus, \odot)$ is a $C^\infty(M)$ module.

Fact 6.5. Under the ZFC set theory axioms, every vector space has a basis. However, there is no such result for modules.

If $\Gamma(TM)$ indeed had a basis, then we could write any vector field as a linear combination of basis vector fields scaled by functions in $C^\infty(M)$. This is however not possible, and we see this through an example by considering a smooth vector field on a sphere. The sphere then must then contain a point where the vector field is zero, where the smooth basis element looses its capability to represent other vector fields.
However, locally over a chart (U, x), we can define the chart induced basis tangent vector field $\frac{\partial}{\partial x^i}$ as

$$\begin{aligned} \frac{\partial}{\partial x^i} \colon U &\to TU \\ p &\mapsto \left(\frac{\partial}{\partial x^i}\right)_p. \end{aligned}$$

6.5 Tensor Fields

We have constructed sections over tangent bundles, which is the set $\Gamma(TM)$. We can similarly construct sections over the cotangent bundle $\Gamma(T^*M)$, which is the set of covector fields. Similarly like $\Gamma(TM)$, we can scale its elements using smooth functions, and therefore it is a $C^\infty(M)$ module.

Definition 6.6 (Tensor field)**.** An (r, s)-tensor field T is a $C^\infty(M)$-multilinear map defined as

$$T\colon \underbrace{\Gamma(T^*M) \times \ldots \times \Gamma(T^*M)}_{r\text{ times}} \times \underbrace{\Gamma(TM) \times \ldots \times \Gamma(TM)}_{s\text{ times}} \xrightarrow{\sim} C^\infty(M).$$

Example 2. For a smooth manifold $(M, \mathcal{O}, \mathcal{A})$, and for $f \in C^\infty(M)$, define the $(0, 1)$ tensor field

$$\begin{aligned} \mathrm{d}f\colon \Gamma(TM) &\xrightarrow{\sim} C^\infty(M) \\ \chi &\mapsto \mathrm{d}f(\chi) := \chi f \end{aligned}$$

where $(\chi f)(p) := \chi(p)f$ for $p \in M$. Also note that $\mathrm{d}f$ is $C^\infty(M)$-linear. This tensor field is called the gradient covector field.

7 Connections

So far, we have seen that a vector field X on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$, can be used to provide a directional derivative Xf of a smooth function $f \in C^\infty(M)$. We define a general notation ∇_X, which for smooth functions on the manifold f is defined as

$$\nabla_X f \coloneqq Xf = (\mathrm{d}f)(X).$$

We have seen in § 6 that a vector field $X \in \Gamma(TM)$ can take in a smooth function $f \in C^\infty(M)$, and return another element $Xf \in C^\infty(M)$ defined as $(Xf)(p) = X(p)f \in \mathbb{R}$ for every $p \in M$. The notation ∇_X can not only just operate on $C^\infty(M)$, but generally can be extended to arbitrary (p, q)-tensor fields to yield (p, q)-tensor fields.

7.1 Directional derivatives of tensor fields

Definition 7.1 (Connection)**.** A *connection* or a *covariant derivative* ∇ on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a map that takes a pair consisting of a vector field X and a (p, q)-tensor field T, and send them to a (p, q)-tensor field $\nabla_X T$, satisfying

1. Extension: For a $(0, 0)$-tensor field, or a smooth function $f \in C^\infty(M)$,

$$\nabla_X f \coloneqq Xf.$$

2. Additivity: For two (p, q) tensors T and S,

$$\nabla_X(T + S) \coloneqq \nabla_X(T) + \nabla_X(S).$$

3. Leibnitz rule: For a (p, q) tensor field T, p covector fields, $\{\omega_i\}_{i=1}^p$ and q vector fields $\{X_j\}_{j=1}^q$, we have

$$\begin{aligned}\nabla_X(T(\omega, Y)) \coloneqq{}& (\nabla_X T)(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q) \\ &+ \sum_{i=1}^{p} T(\ldots, \omega_{i-1}, \nabla_X \omega_i, \omega_{i+1}, \ldots, X_j, \ldots) + \sum_{j=1}^{q} T(\ldots, \omega_i, \ldots, X_{j-1}, \nabla_X X_j, X_{j+1}, \ldots).\end{aligned}$$

4. C^∞-linearity in first argument: For a smooth function $f \in C^\infty(M)$ and a vector field $Z \in \Gamma(TM)$,

$$\nabla_{fX+Z} T \coloneqq f \nabla_X T + \nabla_Z T.$$

A smooth manifold with connection is a quadruple of structures $(M, \mathcal{O}, \mathcal{A}, \nabla)$.

Remark 7.2. The first argument of the connection need not be a vector field defied on the whole manifold. It can also be similarly defined even if the first argument is just a tangent vector at a point.

Remark 7.3. ∇ is not $C^\infty(M)$-linear in the second arguments since we have for two $(0, 0)$-tensor fields $f, g \in C^\infty(M)$, $\nabla_X(fg) = X(fg) = (Xf)g + f(Xg) \neq f(Xg)$.

Remark 7.4. We see that $\nabla_X \cdot$ is the extension of $X \cdot$, and ∇ is the extension of d.

7.2 Structure required to fix a connection

From Definition 7.1 we find that there is still some freedom left to completely specify ∇ over a manifold $(M, \mathcal{O}, \mathcal{A})$. Consider two vector fields $X, Y \in \Gamma(TM)$, and a chart (U, x), then

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) \\ &= X^i \nabla_{\frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) \\ &= X^i \left(\left(\nabla_{\frac{\partial}{\partial x^i}} Y^m \right) \cdot \frac{\partial}{\partial x^m} + Y^m \cdot \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m} \right) \right)\end{aligned}$$

$$= X^i \cdot \left(\nabla_{\frac{\partial}{\partial x^i}} Y^m\right) \cdot \frac{\partial}{\partial x^m} + X^i \cdot Y^m \cdot \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m}\right)$$
$$= X^i \cdot \left(\frac{\partial}{\partial x^i} Y^m\right) \cdot \frac{\partial}{\partial x^m} + X^i Y^m \cdot \Gamma^q_{m,i} \frac{\partial}{\partial x^q}, \tag{7.1}$$

for some coefficients $\Gamma^q_{m,i} \in C^\infty(x^{-1}(U))$. These coefficients are called the *connection coefficient* functions of ∇ with respect to the chart (U, x).

Definition 7.5 (Connection coefficients)**.** For a smooth d dimensional manifold with a connection $(M, \mathcal{O}, \mathcal{A}, \nabla)$ and a chart $(U, x) \in \mathcal{A}$, then the connection coefficients are d^3 many chart dependent functions such that for any $i, j, k \in [d]$

$$\Gamma^i_{j,k} \colon U \to \mathbb{R}$$
$$p \mapsto \left(\mathrm{d}x^i\left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right)\right)(p).$$

Thus, we have from Equation (7.1) the i-th $C^\infty(M)$ coefficient of $\nabla_X Y$ satisfies

$$(\nabla_X Y)^i = X^m \cdot \left(\frac{\partial}{\partial x^m} Y^i\right) + \Gamma^i_{s,m} \cdot Y^s \cdot X^m$$
$$= X(Y^i) + \Gamma^i_{s,m} Y^s X^m. \tag{7.2}$$

Remark 7.6. On a chart domain U, the choice of d^3 functions suffices to fix the action of ∇ on a vector field. The same d^3 functions also fix the action of ∇ on any tensor field.

We can similarly work out the action of ∇_X on a covector field ω, and we will need to determine quantities of the form $\nabla_{\frac{\partial}{\partial x^m}}(\mathrm{d}x^i)$ and represent it as $\Sigma^i_{j,m}\,\mathrm{d}x^j$. Now note that

$$0 = \frac{\partial}{\partial x^m} \delta^i_j$$
$$= \frac{\partial}{\partial x^m}\left(\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)\right)$$
$$= \nabla_{\frac{\partial}{\partial x^m}}\left(\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)\right)$$
$$= \left(\nabla_{\frac{\partial}{\partial x^m}} \mathrm{d}x^i\right)\left(\frac{\partial}{\partial x^j}\right) + (\mathrm{d}x^i)\left(\nabla_{\frac{\partial}{\partial x^m}}\left(\frac{\partial}{\partial x^j}\right)\right)$$
$$= \left(\nabla_{\frac{\partial}{\partial x^m}} \mathrm{d}x^i\right)\left(\frac{\partial}{\partial x^j}\right) + (\mathrm{d}x^i)\left(\Gamma^q_{j,m} \frac{\partial}{\partial x^q}\right)$$
$$\implies \Sigma^i_{j,m} = \left(\nabla_{\frac{\partial}{\partial x^m}} \mathrm{d}x^i\right)_j = -\Gamma^i_{j,m} \tag{7.3}$$

Therefore and similarly as in Equation (7.2), we can again compute the $C^\infty(M)$ coefficients of $\nabla_X \omega$ as

$$(\nabla_X \omega)_i = X(\omega_i) - \Gamma^s_{i,m} \omega_s X^m, \tag{7.4}$$

and the $C^\infty(M)$ coefficients of $\nabla_X T$ for a $(1, 2)$-tensor field T using the Leibnitz rule as

$$(\nabla_X T)^i_{j,k} = X\big(T^i_{j,k}\big) + \Gamma^i_{s,m} T^s_{j,k} X^m - \Gamma^s_{j,m} T^i_{s,k} X^m - \Gamma^s_{k,m} T^i_{j,s} X^m. \tag{7.5}$$

Definition 7.7 (Divergence)**.** Let X be a vector field on a smooth manifold with connection $(M, \mathcal{O}, \mathcal{A}, \nabla)$, then the divergence of X is the function $\operatorname{div} X$ defined as

$$\operatorname{div} X \coloneqq \left(\nabla_{\frac{\partial}{\partial x^i}} X\right)^i,$$

which turns out to be chart independent.

7.3 Change of connection coefficients under change of charts

Consider two charts $(U,x),(V,y)\in\mathcal{A}$ for a smooth manifold with a connection $(M,\mathcal{O},\mathcal{A},\nabla)$, and if $U\cup V\neq\emptyset$, then the following compatibility condition must hold for their respective connections $\Gamma^{i}_{(x),j,k}$ and $\Gamma^{i}_{(y),j,k}$

$$\begin{aligned}
\Gamma^{i}_{(y),j,k} &= \mathrm{d}y^i\left(\nabla_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}\right)\\
&= \frac{\partial y^i}{\partial x^q}\,\mathrm{d}x^q\left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right)\\
&= \frac{\partial y^i}{\partial x^q}\,\mathrm{d}x^q\left(\frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s}+\frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)\right]\right)\\
&= \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\,\mathrm{d}x^q\left[\frac{\partial}{\partial x^p}\left(\frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s}\right]+\frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\,\mathrm{d}x^q\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)\\
&= \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}\left(\frac{\partial x^s}{\partial y^j}\right)\delta^q_s+\frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\Gamma^q_{(x),s,p}\\
&= \frac{\partial y^i}{\partial x^q}\frac{\partial}{\partial y^k}\left(\frac{\partial x^s}{\partial y^j}\right)\delta^q_s+\frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\Gamma^q_{(x),s,p}\\
&= \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{(x),s,p}+\frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j\,\partial y^k}.
\end{aligned}\tag{7.6}$$

From Equation (7.6) we see that even if the connection coefficient functions in one chart are zero, we can still have non-zero connection coefficient functions in the other chart.

7.4 Normal coordinates

Let $p\in M$ be a point in a smooth manifold with connection $(M,\mathcal{O},\mathcal{A},\nabla)$, then one can construct a chart (U,x) such that $p\in U$ and $x(p)=(\alpha^1,\dots,\alpha^d)$ and the connection coefficient functions $\Gamma^i_{(x),(j,k)}$ vanish at p where (j,k) denotes the symmetric part, but not necessarily in any neighborhood. To see this, let (V,y) be any chart with $p\in V$ and $y(p)=0$. Thus in general, $\Gamma^i_{(y),(j,k)}\neq 0$. Then consider a new chart (U,x) to which one transits by virtue of the chart transition map $(x\circ y^{-1})$ defined as

$$(x\circ y^{-1})^i(\alpha^1,\dots,\alpha^d)\coloneqq\alpha^i-\frac{1}{2}\Gamma^i_{(y),(j,k)}(p)\alpha^j\alpha^k.$$

Then,

$$\begin{aligned}
\frac{\partial x^i}{\partial y^j} &= \partial_b\big(x^i\circ y^{-1}\big)\\
&= \delta^i_j-\Gamma^i_{(y),(m,j)}(p)\alpha^m, \qquad \text{and}\\
\frac{\partial^2 x^i}{\partial y^j\,\partial y^k} &= -\Gamma^i_{(y),(k,j)}(p).
\end{aligned}$$

Therefore, on the chart (U,x),

$$\Gamma^i_{(x),j,k}(p)=\Gamma^i_{(y),j,k}(p)-\Gamma^i_{(y),(k,j)}(p)=\Gamma^i_{(y),[j,k]},$$

This gives us that the symmetric part $\Gamma^i_{(x),(j,k)}=0$. Then (U,x) is called a *normal coordinate chart* of the connection ∇ at $p\in M$.

8 Parallel Transport and Curvature

Consider the manifold smooth manifold $(\mathbb{S}^2, \mathcal{O}, \mathcal{A})$. We can imagine this to be of any shape up to diffeomorphisms. Without any extra structure, the round sphere and an ellipsoid are the same objects. If we equip this smooth manifold with an additional structure of connections or define the covariant derivative, only then it takes *curvature*. We will consider the smooth manifold with connection $(M, \mathcal{O}, \mathcal{A}, \nabla)$.

8.1 Parallelity of vector fields

Definition 8.1 (Parallel transport)**.** A vector field X on M is said to be parallely transported along a smooth curve $\gamma\colon \mathbb{R} \to M$ if $\nabla_{v_\gamma} X = 0$, or for any $\lambda \in \mathbb{R}$, we have

$$\left(\nabla_{v_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = 0. \tag{8.1}$$

Remark 8.2. Note that v_γ in Definition 8.1 is not a tangent vector field defined on the whole space, but is only defined on the curve γ, which is consistent with the definition of a connection as per Remark 7.2.

A slightly weaker condition than parallely transported is being *parallel*.

Definition 8.3 (Parallel along a curve)**.** A vector field X is said to be *parallel* along a curve $\gamma\colon \mathbb{R} \to M$ if $\nabla_{v_\gamma} X = \mu \cdot X$, or for all $\lambda \in \mathbb{R}$,

$$\left(\nabla_{v_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = \mu(\lambda) \cdot X_{\gamma(\lambda)},$$

for some $\mu\colon \mathbb{R} \to \mathbb{R}$.

Example 1. For the Euclidean plane $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_{\mathrm{E}})$ consider the vector fields along the curve γ as shown in Figure 9, where the corresponding vector fields are parallely transported, parallel and non-parallel.

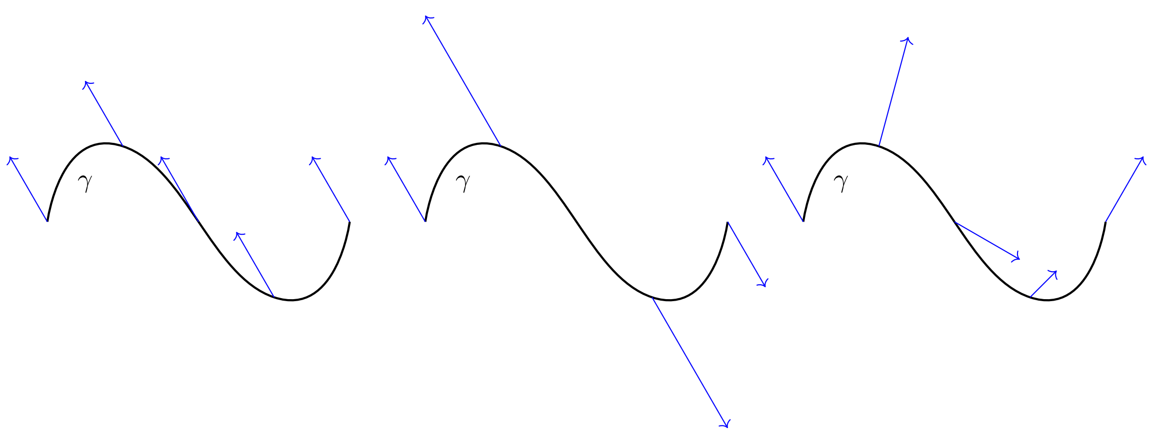


Figure 9: Parallely transported vector field, parallel vector field, and non-parallel vector field

Note that parallel transport is a derived notion from connections.

8.2 Autoparallely transported curves

Definition 8.4. A curve $\gamma\colon \mathbb{R} \to M$ is called *autoparallely transported*, or an *autoparallel curve* if

$$\begin{aligned} \nabla_{v_\gamma} v_\gamma = 0, \quad \text{or} \\ \forall\, \lambda \in \mathbb{R} \qquad \left(\nabla_{v_{\gamma,\gamma(\lambda)}} v_\gamma\right)_{\gamma(\lambda)} = 0. \end{aligned} \tag{8.2}$$

Remark 8.5. We can define a weaker notion of *autoparallel curves* where a curve $\gamma\colon \mathbb{R} \to M$ just satisfies $\nabla_{v_\gamma} v_\gamma = \mu \cdot v_\gamma$

Example 2. For the Euclidean plane $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_{\mathrm{E}})$, consider the curves $\gamma_1, \gamma_2, \gamma_3$ as shown in Figure 10, where they correspond to autoparallely transported, autoparallel and non-autoparallel curves.

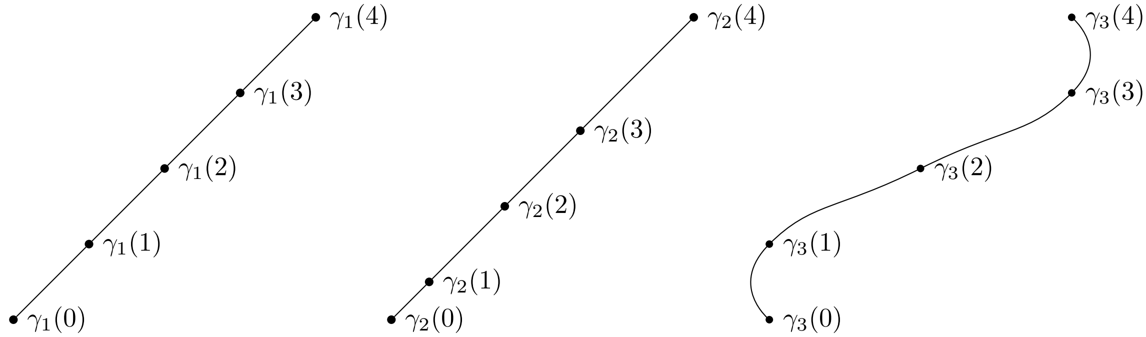


Figure 10: Autoparallely transported, autoparallel and non-autoparallel curves in $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_{\mathrm{E}})$

8.3 Autoparallel equation

Consider a portion of an autoparallely transported curve γ that lies in $(U, x) \in \mathcal{A}$. To express $\nabla_{v_\gamma} v_\gamma = 0$ can be also written in terms of chart representation. Then the tangent field is

$$v_\gamma = \dot{\gamma}^m_{(x)} \cdot \frac{\partial}{\partial x^m}$$

where $\gamma^m_{(x)} \coloneqq x^m \circ \gamma$. Then

$$\begin{aligned} 0 \overset{!}{=} \nabla_{v_\gamma} v_\gamma &= \nabla_{\dot{\gamma}^m_{(x)} \frac{\partial}{\partial x^m}} \dot{\gamma}^n_{(x)} \cdot \frac{\partial}{\partial x^n} \\ &= \dot{\gamma}^m_{(x)} \frac{\partial \dot{\gamma}^n_{(x)}}{\partial x^m} \frac{\partial}{\partial x^n} + \dot{\gamma}^m_{(x)} \dot{\gamma}^n_{(x)} \Gamma^q_{\ n,m} \frac{\partial}{\partial x^q} \\ &= \dot{\gamma}^m_{(x)} \frac{\partial \dot{\gamma}^q_{(x)}}{\partial x^m} \frac{\partial}{\partial x^q} + \dot{\gamma}^m_{(x)} \dot{\gamma}^n_{(x)} \Gamma^q_{\ n,m} \frac{\partial}{\partial x^q} \\ &= \left(\ddot{\gamma}^q_{(x)} + \dot{\gamma}^m_{(x)} \dot{\gamma}^n_{(x)} \Gamma^q_{\ n,m} \right) \frac{\partial}{\partial x^q} \end{aligned}$$

Therefore, for an autoparallel curve, for any $\lambda \in \mathbb{R}$ we have for all m,

$$\ddot{\gamma}^m_{(x)}(\lambda) + \Gamma^m_{(x),a,b}(\gamma(\lambda)) \dot{\gamma}^a_{(x)}(\lambda) \dot{\gamma}^b_{(x)}(\lambda) = 0, \tag{8.3}$$

the chart expression of the condition that γ be autoparallely transported.

Example 3. Consider the Euclidean plane with a chart $(U = \mathbb{R}^2, x = \mathrm{id}_{\mathbb{R}^2})$ and connections $\Gamma^i_{(x),j,k} = 0$. Then the autoparallel equation is $\ddot{\gamma}^m_{(x)} = 0$ for this chart. This takes solutions of the form $\gamma^m_{(x)}(\lambda) = a^m \lambda + b^m$ for $a, b \in \mathbb{R}^2$.

Example 4. For the round sphere $(\mathbb{S}^2, \mathcal{O}, \mathcal{A}, \nabla_{\mathrm{round}})$, consider a chart $(U, x = (\theta, \varphi))$, where $U = x^{-1}((0, \pi) \times (0, 2\pi))$, and the connections on the chart

$$\begin{pmatrix} \Gamma^1_{(x),1,1} & \Gamma^1_{(x),1,2} \\ \Gamma^1_{(x),2,1} & \Gamma^1_{(x),2,2} \end{pmatrix} \left(x^{-1}(\theta, \varphi) \right) = \begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix}, \qquad \text{and}$$
$$\begin{pmatrix} \Gamma^2_{(x),1,1} & \Gamma^2_{(x),1,2} \\ \Gamma^2_{(x),2,1} & \Gamma^2_{(x),2,2} \end{pmatrix} \left(x^{-1}(\theta, \varphi) \right) = \begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}.$$

Then the autoparallel Equation (8.3) becomes

$$\begin{aligned} \ddot{\theta} - \sin\theta\cos\theta \ \dot{\varphi}\dot{\varphi} = 0, \qquad &\text{and} \\ \ddot{\varphi} + 2\cot\theta \ \dot{\theta}\dot{\varphi} = 0. \end{aligned}$$

Note that $\theta(\lambda) = 0$ and $\varphi(\lambda) = \omega \cdot \lambda + \varphi_0$ for some $\omega, \varphi_0 \in \mathbb{R}$ is a solution to the autoparallel equation on the round sphere.

Remark 8.6. Autoparallel curves are the *straightest* curves on a manifold, and does not need the notion of being the fastest as there is no metric yet defined on the manifold.

8.4 Torsion

Definition 8.7 (Torsion)**.** A torsion T of a connection ∇ is a $(1,2)$-tensor field such that for a covector field $\omega \in \Gamma(T^*M)$ and two vector fields $X, Y \in \Gamma(TM)$,

$$T(\omega, X, Y) \coloneqq \omega(\nabla_X Y - \nabla_Y X - [X, Y]), \tag{8.4}$$

where $[X, Y]$ is defined as $[X, Y]f \coloneqq X(Yf) - Y(Xf)$ for $f \in C^\infty(M)$.

Observe that T is antisymmetric in the 2^{nd} and 3^{rd} argument. It remains to check that T is $C^\infty(M)$ linear in its arguments. Let $f \in C^\infty(M)$, $\psi \in \Gamma(T^*M)$ then

$$\begin{aligned}
T(f \cdot \omega, X, Y) &= f \cdot \omega(\nabla_X Y - \nabla_Y X - [X, Y]) = f \cdot T(\omega, X, Y), \\
T(\omega + \psi, X, Y) &= \omega(\nabla_X Y - \nabla_Y X - [X, Y]) + \psi(\nabla_X Y - \nabla_Y X - [X, Y]) \\
&= T(\omega, X, Y) + T(\psi, X, Y) \\
T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]) \\
&= \omega(f\nabla_X Y - ((Yf)X + f\nabla_Y X) - [fX, Y]).
\end{aligned}$$

The term $[fX, Y]$ can be simplified by observing its action on some $g \in C^\infty(M)$, as

$$\begin{aligned}
[fX, Y]g &= f \cdot X(Yg) - Y(f \cdot (Xg)) \qquad \text{(product rule)} \\
&= fX(Yg) - (Yf)(Xg) - fY(Xg) \\
&= f[X, Y]g - (Yf)Xg \\
\implies [fX, Y] &= f[X, Y] - (Yf)X,
\end{aligned}$$

using which we get

$$\begin{aligned}
T(\omega, fX, Y) &= \omega(f\nabla_X Y - ((Yf)X + f\nabla_Y X) - f[X, Y] + (Yf)X) \\
&= \omega(f\nabla_X Y - f\nabla_Y X - f[X, Y]) \\
&= f \cdot T(\omega, X, Y), \\
\implies T(\omega, X, fY) &= -T(\omega, fY, X) = -f \cdot T(\omega, Y, X) = f \cdot T(\omega, X, Y).
\end{aligned}$$

In a similar way, using the definition of connection and the bracket $[\cdot, \cdot]$, we can show that T is additive linear in the 2^{nd} and 3^{rd} arguments.

Definition 8.8 (Torsion-free)**.** A smooth manifold with connection $(M, \mathcal{O}, \mathcal{A}, \nabla)$ is called torsion-free if $T = 0$. That is, in a chart (U, x), we have

$$\begin{aligned}
T^i_{a,b} &\coloneqq T\left(\mathrm{d}x^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) \\
&= \mathrm{d}x^i\left(\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} - \nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^a} - \left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right]\right) \\
&= \Gamma^i_{a,b} - \Gamma^i_{b,a} = 2\Gamma^i_{[a,b]}.
\end{aligned}$$

Form hereon, our focus will be on torsion-free connections unless specified otherwise.

8.5 Curvature

Definition 8.9 (Riemann curvature)**.** The *Riemann curvature* Riem of a connection ∇ is the $(1,3)$-tensor field such that for $\omega \in \Gamma(T^*M)$ and $Z, X, Y \in \Gamma(TM)$ is defined as

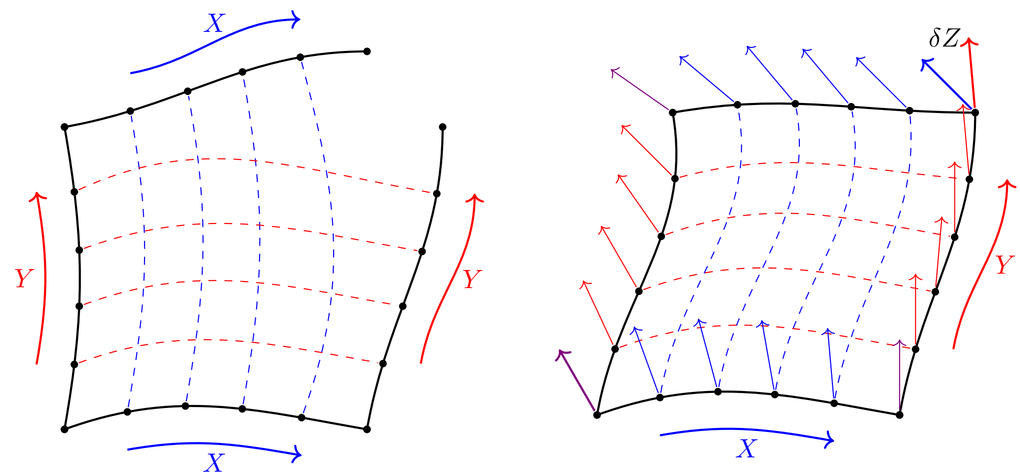
$$\mathrm{Riem}(\omega, Z, X, Y) \coloneqq \omega\big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\big).$$

In any number of dimension, the Riemann curvature with 4 indices is sufficient to contain all the information about the notion of curvature. In a chart (U, x), if $\nabla_q \equiv \nabla_{\frac{\partial}{\partial x^q}}$, for all q, and if $a \neq b$, then

$$(\nabla_a \nabla_b Z)^m - (\nabla_b \nabla_a Z)^m = \mathrm{Riem}^m_{n,a,b}\, Z^n + \nabla_{\left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right]} Z$$

$$= \text{Riem}^m_{n,a,b}\, Z^n,$$

and the Riemann tensor components contain all the information of how the covariant derivatives fail to commute if they act on a vector field. Or in other words, we can only swap the order of covariant derivatives if the Riemann curvature is zero. Because the Riemann tensor is a tensor, if it is zero in one coordinate system, it is also zero in other coordinate systems.



(a) Example when $[X, Y] \neq 0$.

(b) Example when $[X, Y] = 0$

Figure 11: Effect of curvature on parallel transport

For a torsion-free manifold, for example in Figure 11b, we see that the difference caused to the action of the vector field X and Y in different order is non-zero, and can be shown to be equal to

$$(\delta Z)^m = \text{Riem}^m_{n,a,b}\, X^a Y^b Z^n \delta_s \delta_t + \mathcal{O}\big(\delta_s^2 \delta_t, \delta_t^2 \delta_s\big),$$

where δ_s and δ_t are the parameter distances of curves parallel to X and Y respectively. The difference in the action of two vector fields one after the other will be zero in the limit if the curvature of the manifold is zero.

Another notion of curvature called the *Ricci curvature tensor*.

Definition 8.10 (Ricci curvature tensor)**.** The *Ricci curvature tensor* is the contraction of the Riemann curvature tensor with respect to the first and the third indices, i.e., $R_{a,b} = \text{Riem}^c_{a,c,b}$.

9 Newtonian Spacetime is Curved

Newton's first two axioms of motion can be written down as follows:

1. A body on which *no* force acts moves uniformly along a *straight line*.
2. Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.

Remark 9.1. The first axiom in order to be relevant must be read as a measurement prescription for the *geometry* of space.

Remark 9.2. Since gravity universally acts on every particle in a universe with at least two particles, gravity must not be considered a force if the first axiom is supposed to remain applicable.

Newton's axioms of motion are therefore under a serious question, first asked by Laplace.

Laplace's question: Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of no other force are postulated to move along straight lines in this curved space?
The answer to Laplace's question is however, a *no*. To see this, let us consider the force field point of view of gravity, using which we can write down the equation,

$$\ddot{x}^\alpha(t) = f^\alpha(x(t)), \tag{9.1}$$

where x is the position coordinate of a non-zero mass particle, and f is the gravitational force field. Laplace's question can be interpreted by asking whether there exists a connection Γ on the geometry of space such that

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0 \qquad \equiv \quad \ddot{x}^\alpha(t) + \Gamma^\alpha_{\beta,\gamma}\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0,$$

where the former equation is an autoparallel equation (refer Equation (8.3)). We see that it is not possible to make the two equations equivalent because the former does not depend on the first derivatives of the position. Therefore, one cannot find Γ such that Newton's equation takes the form of an autoparallel equation.

9.1 The full wisdom of Newton's first axiom of motion

To successfully implement Laplace's idea of finding a curvature of space such that particles with mass move along straight lines, we need to revisit Newton's first axiom which talks about *uniform motion* along a straight line, and not just motion along a straight line. Therefore we use the information from Newton's first axiom of motion that particles under the influence of no force move uniformly. Refer to Figure 10 to exhibit the meaning of uniform motion and non-uniform motion. In *spacetime*, 'straight motion' is simply 'uniform and straight motion' in space. Therefore, we can ignore the parameterization of a curve on space using time, by storing the information in an extra coordinate. Therefore, Laplace's idea might work, but in 'Newtonian' spacetime rather than just space.
Let $x\colon \mathbb{R} \to \mathbb{R}^3$ be a particle's trajectory in space. We can convert this information into a 'worldline' of the particle defined as

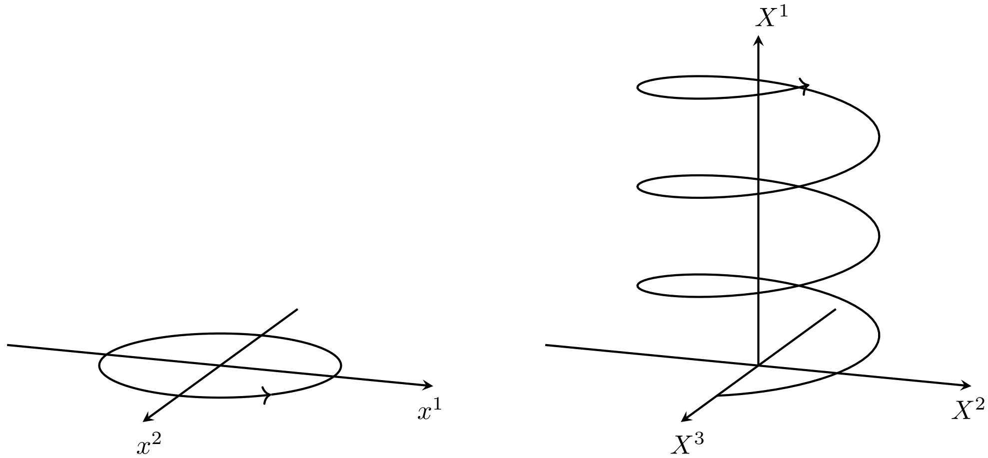
$$\begin{aligned} X\colon \mathbb{R} &\to \mathbb{R}^4 \\ t &\mapsto (t, x^1(t), x^2(t), x^3(t)), \end{aligned}$$

as shown in Figure 12.
Let us assume that x satisfies the Newtonian gravitational field Equation (9.1), i.e.,

$$\ddot{x}^\alpha(t) = f^\alpha(x(t)). \tag{9.2}$$

A trivial re-writing of Equation (9.2) with respect to the coordinate system X would be therefore $\dot{X}^0 = 1$ along with

$$\begin{aligned} \ddot{X}^0 &= 0, \\ \ddot{X}^\alpha - f^\alpha(X(t)) &= 0 \qquad \forall\ \alpha \in [3], \\ \implies \ddot{X}^\alpha - f^\alpha(X(t))\dot{X}^0\dot{X}^0 &= 0 \qquad \forall\ \alpha \in [3]. \end{aligned} \tag{9.3}$$



(a) Trajectory in space

(b) Worldline in spacetime

Figure 12: Equivalence in x and X parameterization

Equations (9.3) is equivalent to the following for all $a \in \{0\} \cup [3]$

$$(\nabla_{v_X} v_X)^a = \ddot{X}^a + \Gamma^a_{b,c}\dot{X}^b\dot{X}^c = 0, \tag{9.4}$$

which is the autoparallel equation for Γ defined as $\Gamma^0_{a,b} = 0$ for all $a, b \in \{0\} \cup [3]$, $\Gamma^\alpha_{0,0} \overset{!}{=} -f^\alpha$ for $\alpha \in [3]$, and all the rest set to 0. Since we chose the standard Euclidean chart, therefore it is left to check whether this is a coordinate choice artifact. We can see that this is not the case by computing the Riemann curvature which only on the non-vanishing components will be $\mathrm{Riem}^\alpha_{0,\beta,0} = -\frac{\partial}{\partial x^\beta} f^\alpha$ for $\alpha, \beta \in [3]$. This from Definition 8.10 also gives us the Ricci curvature tensor which for the non-vanishing components is $R_{0,0} = -\partial_\alpha f^\alpha$. On the other the Poisson's gravity equation from Newtonian gravitational theory hand tells us $-\partial_\alpha f^\alpha = 4\pi G\rho$, where G is the universal gravitational constant, and ρ is the mass density. Therefore, we see that the $(0,0)$ component of the Ricci curvature tensor of Newtonian spacetime is $R_{0,0} = 4\pi G\rho$.

Remark 9.3. The Newtonian space time Riemann curvature tensor is 0 in the space coordinates, i.e., $\mathrm{Riem}^\alpha_{\beta,\gamma,\delta} = 0$ for all $\alpha, \beta, \gamma, \delta \in [3]$. The curvature is only present in the extra time direction together with the spacial direction.

Definition 9.4 (Acceleration vector)**.** The covariant derivative $\nabla_{v_X} v_X$ is called the *acceleration vector* $\mathfrak{a}$.

Therefore, Laplace's idea works on Newtonian spacetime. We can now reformulate Newtonian gravity as a curvature of spacetime, and the first axiom takes the form: Particles under the influence of no force has a worldline (in spacetime) that is straight.

9.2 The foundations of the geometric formulation of Newton's axioms

Definition 9.5 (Newtonian spacetime)**.** A *Newtonian spacetime* is quintuple $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$, where $(M, \mathcal{O}, \mathcal{A})$ is a 4 dimensional smooth manifold, and $t \colon M \to \mathbb{R}$ is a smooth function satisfying

1. There is absolute space, i.e., $(\mathrm{d}t)_p \neq 0$ for all $p \in M$, and the absolute space at time τ is the set $S_\tau := \{p \in M \mid t(p) = \tau\}$. This leads to writing the entire manifold as $M = \dot{\bigcup}_\tau S_\tau$.
2. Absolute time flows uniformly, i.e., the $(0,2)$-tensor field $\nabla \mathrm{d}t = 0$ everywhere.
3. ∇ is torsion-free.

Definition 9.6 (Directed tangent vectors)**.** A vector $X \in T_pX$ for a point $p \in M$ is called

1. *future-directed* if $\mathrm{d}t(X) > 0$,
2. *spatially-directed* if $\mathrm{d}t(X) = 0$, and
3. *past-directed* if $\mathrm{d}t(X) < 0$.

With the above definition, Newton's first axiom of motion can be reformulated as: The worldline of a particle under the influence of no force, is a future-directed autoparallel curve. Therefore, the tangent vector X of a worldline of a particle satisfies $\nabla_{v_X} v_X = 0$, and $\mathrm{d}t(X) > 0$. Newton's second axiom of motion can be reformulated as: $\nabla_{v_X} v_X = \frac{F}{m}$, where F is a spacial vector field, i.e., $\mathrm{d}t(F) = 0$, or equivalently $m \cdot \mathfrak{a} = F$.

As a convention, we will restrict ourselves to atlases $\mathcal{A}_{\text{stratified}}$ whose charts (U, x) have the property $x^0 = t\big|_U$. Using this convention, we can re-write the second point in Definition 9.5 as

$$0 = \left(\nabla_{\frac{\partial}{\partial x^a}} \mathrm{d}x^0\right)_b = -\Gamma^0_{b,a}, \tag{9.5}$$

for all $a, b \in \{0\} \cup [3]$.
Consider a chart (U, x) in the stratified atlas $\mathcal{A}_{\text{stratified}}$, the Newton's second axiom of motion for an external force field F says

$$\nabla_{v_X} v_X = \frac{F}{m},$$

which on the chart (U, x), and under any arbitrary parameterization via $\lambda \in \mathbb{R}$ can be written as

$$X^{0''} + \Gamma^0_{c,d} X^{c'} X_d{}' = 0,$$
$$X^{\alpha''} + \Gamma^\alpha_{\gamma,\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{0,0} X^{0'} X^{0'} + \Gamma^\alpha_{\gamma,0} X^{\gamma'} X^{0'} + \Gamma^\alpha_{0,\gamma} X^{0'} X^{\gamma'} = \frac{F^\alpha}{m},$$

for $c, d \in \{0\} \cup [3]$, and $\alpha, \gamma, \delta \in [3]$. Since we are on a chart of a stratified atlas, and the connection is torsion-free, we can use Equation (9.5) to obtain $X^{0''} = 0$, which solves as $X^0 = a\lambda + b$ for $a, b \in \mathbb{R}$, or $(x^0 \circ X)(\lambda) = (t \circ X)(\lambda) = a\lambda + b$, and therefore

$$\mathfrak{a}^a := \ddot{X}^\alpha + \Gamma^\alpha_{\gamma,\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{0,0} + 2\Gamma^\alpha_{\gamma,0} \dot{X}^\gamma = \frac{1}{a^2} \frac{F^\alpha}{m},$$

is the chart coordinates of the acceleration field $\mathfrak{a}$. The extra a^2 on th right is due to the change in scale of measurement of time. In the presence of gravity, we have $\Gamma^\alpha_{0,0} = -f^\alpha$, else 0. The term $\Gamma^\alpha_{\gamma,\delta}$ which is the connection with only the spacial indices, is the correction terms when the spacial coordinates of the chart are not Euclidean (e.g., polar coordinates). Now consider a rotating coordinate system in space, then the term $\Gamma^\alpha_{0,0}$ corresponds to the centrifugal pseudo-acceleration, which is independent of velocity, but shall depend on the angular frequency of the chart. The term $2\Gamma^\alpha_{\gamma,0} \dot{X}^\gamma$ corresponds to the Coriolis pseudo-acceleration is proportional to the spacial velocity of the particle, and shall also depend on the angular frequency of the rotating chart. The acceleration field as we see is independent of the coordinate system we choose, and the terms which make up the total acceleration field is a sum of quantities which add up in ways depending on the coordinate system chosen.

10 Metric Manifolds

We establish a structure on a smooth manifold that allows to assign vectors in each tangent space, a *length* (and angle between vectors in the same tangent space). From this structure, one can then define a notion of length of a curve. Then we can look at shortest curves. Requiring that the shortest curves coincide with the straight curves with respect to some connection ∇ will result in ∇ being determined by the metric structure in the presence of zero torsion. Since the connections fully determine the Riemann curvature, we have that a metric is sufficient to determine curvature.

10.1 Metrics

Definition 10.1 (Metric)**.** A *metric* g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying

1. Symmetry: $g(X,Y) = g(Y,X)\ \forall\ X, Y \in \Gamma(TM)$,
2. Non-degeneracy: the musical map

$$\begin{aligned} \flat\colon \Gamma(TM) &\to \Gamma(T^*M) \\ X &\mapsto \flat(X), \end{aligned}$$

where $\flat(X)(Y) \coloneqq g(X,Y) \in C^\infty(M)\ \forall\ Y \in \Gamma(TM)$, is an isomorphism.

Definition 10.2. The $(2,0)$-tensor field g^{-1} (this is just a notation and must not to be confused by the functional inverse) with respect to a metric g is the symmetric $(2,0)$-tensor field

$$\begin{aligned} g^{-1}\colon \Gamma(T^*M) \times \Gamma(T^*M) &\xrightarrow{\sim} C^\infty(M) \\ (\omega, \sigma) &\mapsto \omega\big(\flat^{-1}(\sigma)\big). \end{aligned}$$

Remark 10.3. The components of $\flat(X) \in \Gamma(T^*M)$ for $X \in \Gamma(TM)$ satisfy $(\flat(X))_a \coloneqq g_{a,m} X^m$. Its inverse $\sharp \coloneqq \flat^{-1}$ for $\omega \in \Gamma(T^*M)$ satisfies $(\sharp(\omega))^a = (\flat^{-1}(\omega))^a \coloneqq \left(g^{-1}\right)^{a,m} \omega_m$.

Remark 10.4. On a chart however, the two conditions in Definition 10.1 are equivalent to $g_{a,b} = g_{b,a}$, and $\left(g^{-1}\right)^{a,m} g_{m,b} = \delta^a_b$.

Example 1. Consider the smooth manifold $(\mathbb{S}^2, \mathcal{O}, \mathcal{A})$ and a chart $(U, x = (\theta, \varphi))$, where $U = x^{-1}((0,\pi) \times (0, 2\pi))$. Define the metric $g_{i,j}\big(x^{-1}(\theta,\varphi)\big) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix}_{i,j}$, of the round sphere of radius $R \in \mathbb{R}_+$.

10.2 Signature

In linear algebra, if we have a $(1,1)$-tensor A, then we have the eigenvalue equation $A^a_m v^m = \lambda v^a$, for eigenvalue λ and eigenvector v. However, it does not make sense for the $(0,2)$-tensor $g_{a,m}$ to have eigenvectors, since its application on a vector does not give back a vector. A $(1,1)$ tensor has eigenvalues, but a $(0,2)$ tensor doesn't. A $(0,2)$ tensor has a signature. For the metric tensor g (with its musical map invertible), its signature is a well-defined (p,q) tuple of ± 1, with p ones and q negative ones. For a d dimensional manifold, the number of possibilities of a signature up to permutation is therefore $d+1$ since $q = d - p$.

Definition 10.5 (Riemannian, Lorentzian and pseudo-Riemannian metric)**.** A metric g is called a *Riemannian metric*, if its signature has all positive ones. A metric g is called a *Lorentzian metric* if its signature has exactly one positive one, and rest negative ones. Metrics with any other signature is called a *pseudo-Riemannian metric.*

10.3 Length of a curve

Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve on a Riemannian manifold $(M, \mathcal{O}, \mathcal{A}, g)$, then we know its velocity at each $\gamma(\lambda) \in M$, for $\lambda \in \mathbb{R}$.

Definition 10.6 (Speed of a curve)**.** On a Riemannian metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$, the *speed* of a curve at $\gamma(\lambda) \in M$ for $\lambda \in \mathbb{R}$ is the number

$$s(\lambda) \coloneqq \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}.$$

Remark 10.7. In arbitrary coordinate charts of a manifold, there is no notion of a measurement units that we can prescribe to the components of a tangent vector. Measurement units get derived through the metric which is embedded on a manifold. The dimension of the coordinates of a tangent (velocity) vector is T^{-1}. On a chart of a metric manifold, the dimension of the coordinates $g_{a,b}$ of the metric g are L^2, which therefore provides us with the dimension of speed to be LT^{-1}. In other words, the idea that coordinate distance has anything to do with real distance is false.

Definition 10.8 (Length of a curve)**.** Let $\gamma\colon (0,1) \to M$ is a smooth curve, then the length of γ is the number

$$L[\gamma] \coloneqq \int_0^1 \mathrm{d}\lambda \; s(\lambda) = \int_0^1 \mathrm{d}\lambda \; \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}. \tag{10.1}$$

Example 2. Consider the round sphere of radius R as in Example 1, and consider the equator $(x_1 \circ \gamma)(\lambda) =: \theta(\lambda) = \pi/2$, $(x_1 \circ \gamma)(\lambda) =: \varphi(\lambda) = 2\pi\lambda^3$. Then, the length of the equatorial curve is

$$\begin{aligned} L[\gamma] &= \int_0^1 \mathrm{d}\lambda \; \sqrt{g_{i,j}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} \\ &= \int_0^1 \mathrm{d}\lambda \; \sqrt{R^2 \sin^2(\pi/2) \cdot 36\pi^2\lambda^4} \\ &= 6\pi R \int_0^1 \mathrm{d}\lambda \; \lambda^2 = 2\pi R. \end{aligned}$$

Theorem 10.9. *If $\gamma\colon (0,1) \to M$, and $\sigma\colon (0,1) \to (0,1)$ is a smooth, bijective and an increasing function, then $L[\gamma] = L[\gamma \circ \sigma]$.*

10.4 Geodesics

Definition 10.10. A curve $\gamma\colon (0,1) \to M$ is called a *geodesic* on a Riemannian manifold $(M, \mathcal{O}, \mathcal{A}, g)$ if it is a stationary curve with respect to the length functional L.

Theorem 10.11. *A curve $\gamma\colon (0,1) \to M$ on a Riemannian manifold $(M, \mathcal{O}, \mathcal{A}, g)$ if and only if it satisfies the Euler-Lagrange equations for the Lagrangian*

$$\begin{aligned} \mathcal{L}\colon TM &\to \mathbb{R} \\ X &\mapsto \sqrt{g(X,X)}. \end{aligned}$$

In a chart (U, x), the Euler-Lagrange equation takes the form

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m}\right)^{\boldsymbol{\cdot}} - \frac{\partial \mathcal{L}}{\partial x^m} = 0.$$

For the given Lagrangian where $\mathcal{L}(\gamma^m, \dot\gamma^m) = \sqrt{g_{i,j}(\gamma^m(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}$, the Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot\gamma^m} &= \frac{2}{2\sqrt{g_{i,j}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}} g_{m,j}(\gamma(\lambda))\dot\gamma^j(\lambda) \\ \implies \left(\frac{\partial \mathcal{L}}{\partial \dot\gamma^m}\right)^{\boldsymbol{\cdot}} &= \left(\frac{1}{\sqrt{g_{i,j}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}}\right)^{\boldsymbol{\cdot}} g_{m,j}(\gamma(\lambda))\dot\gamma^j(\lambda) \\ &\quad + \frac{1}{\sqrt{g_{i,j}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}} \left(g_{m,j}(\gamma(\lambda))\ddot\gamma^j(\lambda) + \dot\gamma^s(\lambda)\,\partial_s g_{m,j}(\gamma(\lambda))\dot\gamma^j(\lambda)\right) \end{aligned}$$

To evaluate the first term, we can use Theorem 10.9 to put a constraint on γ that $g(\dot\gamma, \dot\gamma) = 1$. Then, the first term vanishes under this parameterization. For the second term in the Euler-Lagrange equation, we have

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{g_{i,j}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}} \, \partial_m g_{i,j}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)$$

Putting it all together in the Euler-Lagrange Equation, we get

$$g_{m,j}(\gamma(\lambda))\ddot{\gamma}^j(\lambda) + \partial_s g_{m,j}(\gamma(\lambda))\dot{\gamma}^s(\lambda)\dot{\gamma}^j(\lambda) - \frac{1}{2}\,\partial_m g_{i,j}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda) = 0$$

We can multiply the above Equation with $\left(g^{-1}\right)^{q,m}(\gamma(\lambda))$ for some index q, and sum over m, to obtain

$$\begin{aligned}
0 &= \left(g^{-1}\right)^{q,m}(\gamma(\lambda))\,\partial_s g_{m,j}(\gamma(\lambda))\dot{\gamma}^s(\lambda)\dot{\gamma}^j(\lambda) - \frac{1}{2}\left(g^{-1}\right)^{q,m}(\gamma(\lambda))\,\partial_m g_{i,j}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda) \\
&\quad + \left(g^{-1}\right)^{q,m}(\gamma(\lambda))g_{m,j}(\gamma(\lambda))\ddot{\gamma}^j(\lambda) \\
0 &= \ddot{\gamma}^q(\lambda) + \left(g^{-1}\right)^{q,m}(\gamma(\lambda))\,\partial_i g_{m,j}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda) - \frac{1}{2}\left(g^{-1}\right)^{q,m}(\gamma(\lambda))\,\partial_m g_{i,j}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda) \\
0 &= \ddot{\gamma}^q(\lambda) + \left(g^{-1}\right)^{q,m}(\gamma(\lambda))\Big(\partial_i g_{m,j}(\gamma(\lambda)) - \frac{1}{2}\,\partial_m g_{i,j}(\gamma(\lambda))\Big)\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda) \\
\implies 0 &= \ddot{\gamma}^q + \left(g^{-1}\right)^{q,m}\frac{1}{2}(\partial_i g_{m,j} + \partial_j g_{m,i} - \partial_m g_{i,j})\dot{\gamma}^i\dot{\gamma}^j
\end{aligned}$$

The Equation

$$\ddot{\gamma}^q + \frac{1}{2}\left(g^{-1}\right)^{q,m}(\partial_i g_{m,j} + \partial_j g_{m,i} - \partial_m g_{i,j})\dot{\gamma}^i\dot{\gamma}^j = 0 \tag{10.2}$$

is called the *geodesic equation* for components of γ in a chart. Observe that we can relate Equation (10.2) with Equation (8.3) and define a connection ∇ via the symbols

$${}^{\mathrm{LC}}\Gamma^q_{i,j} \coloneqq \frac{1}{2}\left(g^{-1}\right)^{q,m}(\partial_i g_{m,j} + \partial_j g_{m,i} - \partial_m g_{i,j}). \tag{10.3}$$

Definition 10.12 (Christoffel symbols and Levi-Civita connection)**.** The connection coefficients ${}^{\mathrm{LC}}\Gamma$ as defined in Equation (10.3) are called the *Christoffel symbols* of the *Levi-Civita connection* ${}^{\mathrm{LC}}\nabla$.

By choosing the connection coefficients as in Equation (10.3), we have insisted that the geodesic Equation (10.2) (a property of the metric) be also the autoparallel Equation (8.3) (a property of the connection).
We usually choose this choice of connection if a metric g is given. That is, if we have a smooth manifold with a metric $(M, \mathcal{O}, \mathcal{A}, g)$, we construct from it a smooth manifold with a metric and a connection $(M, \mathcal{O}, \mathcal{A}, g, {}^{\mathrm{LC}}\nabla)$, such that the connection ${}^{\mathrm{LC}}\nabla$ is derived from the metric g. The same construction can be done in a way that is chart independent, if we seek to find a connection ∇ such that $\nabla g = 0$, and its torsion $T = 0$. The solution to this also turns out to be the Levi-Civita connection ${}^{\mathrm{LC}}\nabla$.

Definition 10.13 (Riemann-Christoffel curvature)**.** The *Riemann-Christoffel curvature* R on a metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$ is a $(0, 4)$ tensor is defined as $R_{a,b,c,d} \coloneqq g_{a,m}\,\mathrm{Riem}^m_{b,c,d}$, where the Riemann curvature is defined through the Levi-Civita connection ${}^{\mathrm{LC}}\nabla$.

Definition 10.14 (Ricci tensor)**.** The *Ricci tensor* can be defined without a metric as $R_{a,b} \coloneqq \mathrm{Riem}^m_{a,m,b}$, however if we are on a metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$, the Riemannian curvature tensor is the one which is defined through the Levi-Civita connection ${}^{\mathrm{LC}}\nabla$.

Definition 10.15 (Ricci scalar curvature)**.** The *Ricci scalar curvature* is defined for a metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$ as $R \coloneqq g^{a,b}R_{a,b}$.

As a convention, we denote $\left(g^{-1}\right)^{a,b}$ as simply $g^{a,b}$.

Definition 10.16 (Einstein's curvature)**.** The *Einstein's curvature* tensor is defined on a metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$ as $G_{a,b} \coloneqq R_{a,b} - \frac{1}{2}g_{a,b}R$.

11 Symmetry

With the introduction of a metric on a manifold, we can now talk about 'symmetry' that it induces on the same. Before introducing symmetry formally, we will need to build some more base in differential geometry.

11.1 Push-forward map

Definition 11.1 (Push-forward map)**.** Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ be two smooth manifolds, and let $\phi\colon M \to N$ be a smooth map between the two. Then the *push-forward* map ϕ_* is the map defined as

$$\begin{aligned} \phi_*\colon TM &\to TN, \\ X &\mapsto \phi_*(X), \end{aligned}$$

where for any $f \in C^\infty(N)$, $\phi_*(X)f \coloneqq X(f \circ \phi)$.

$$\begin{array}{ccccc} TM & \xrightarrow{\phi_*} & TN & & \\ {\scriptstyle \pi_{TM}}\downarrow & & \downarrow{\scriptstyle \pi_{TN}} & & \\ M & \xrightarrow{\phi} & N & \xrightarrow{f} & \mathbb{R} \end{array}$$

Note that $\phi_*(T_pM) \subseteq T_{\phi(p)}N$. The components of ϕ_* with respect to two charts, $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$ at a point $p \in U$, can be computed through its action on the tangent bundle as

$$\begin{aligned} {\phi_*}^a_i &\coloneqq \mathrm{d}y^a\left(\phi_*\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)\right) \\ &= \phi_*\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)y^a \\ &= \left(\frac{\partial}{\partial x^i}\right)_p (y^a \circ \phi) \\ &= \left(\frac{\partial}{\partial x^i}\right)_p (y \circ \phi)^a. \end{aligned} \tag{11.1}$$

Let us define the function $(y \circ \phi)^a\colon M \to \mathbb{R}$ as $\hat{\phi}^a$, then ${\phi_*}^a_i = \left(\frac{\partial \hat{\phi}^a}{\partial x^i}\right)_p$. The components ${\phi_*}^a_i$ however are not to be confused with the components of some tensor, since there are two different manifolds involved in its characterization.

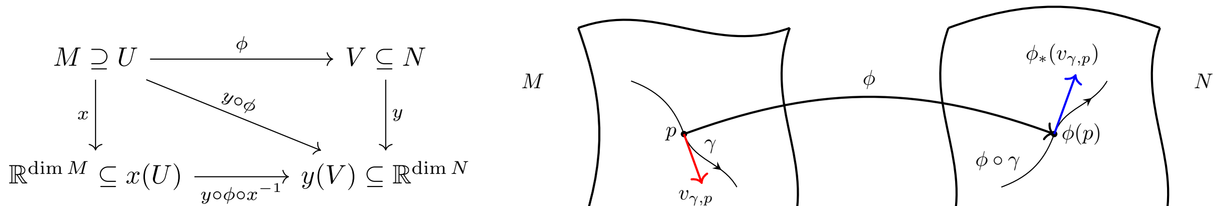


Figure 13: The push-forward map of $\phi\colon M \to N$.

As we see in Figure 13 for example, let for some $\lambda_0 \in \mathbb{R}$, $\gamma(\lambda_0) = 0$, and observe for any $f \in C^\infty(N)$,

$$\begin{aligned} \phi_*(v_{\gamma,p})f &= v_{\gamma,p}(f \circ \phi) \\ &= ((f \circ \phi) \circ \gamma)'(\lambda_0) \qquad \text{(Using Equation (5.1))} \\ &= (f \circ (\phi \circ \gamma))'(\lambda_0) \\ &= v_{\phi\circ\gamma,\phi(p)}f \\ \implies \phi_*(v_{\gamma,p}) &= v_{\phi\circ\gamma,\phi(p)}. \end{aligned} \tag{11.2}$$

11.2 Pull-back map

Definition 11.2 (Pull-back map)**.** Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ be two smooth manifolds, and let $\phi\colon M \to N$ be a smooth map between the two. Then the *pull-back* map ϕ^* is the map defined as

$$\begin{aligned}\phi^*\colon T^*N &\to T^*M\\ \omega &\mapsto \phi^*(\omega),\end{aligned}$$

where for a vector $X \in TM$, $\phi^*(\omega)(X) := \omega(\phi_*(X))$.

The components of the pull-back with respect to two charts, $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$ at a point $p \in U$,

$$\begin{aligned}{\phi^*}^a_{\ i} &:= \phi^*\big((\mathrm{d}y^a)_{\phi(p)}\big)\left(\frac{\partial}{\partial x^i}\right)_p\\ &= (\mathrm{d}y^a)_{\phi(p)}\left(\phi_*\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)\right)\\ &= {\phi_*}^a_{\ i},\end{aligned} \tag{11.3}$$

where the last equality is using Equation 11.1. Therefore, we have for a tangent vector X, $(\phi_*(X))^a = {\phi_*}^a_{\ i} X^i$, and for a cotangent vector ω, $(\phi^*(\omega))^a = {\phi^*}^a_{\ i}\omega^i$

As an application, consider an injective map ϕ between two smooth manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}, \mathcal{A})$ such that $\dim M < \dim N$, and where M is embedded in N, i.e.,

$$M \xhookrightarrow[\text{injective}]{\phi} N.$$

Consider a metric g on the smooth manifold N. Then the induced metric g_M can be defined on every point $p \in M$ and $X, Y \in T_pM$ as

$$\begin{aligned}g_M(X, Y) &:= g(\phi_*(X), \phi_*(Y))\\ \implies \left((g_M)_{i,j}\right)_p &= (g_{a,b})_{\phi(p)}\left(\frac{\partial\hat{\phi}^a}{\partial x^i}\right)_{\phi(p)}\left(\frac{\partial\hat{\phi}^b}{\partial x^i}\right)_{\phi(p)}.\end{aligned} \tag{11.4}$$

11.3 Flow of a complete vector field

Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold, and $X \in \Gamma(TM)$.

Definition 11.3 (Integral curve)**.** For an interval $I \in \mathbb{R}$, a curve $\gamma\colon I \to M$ is called an *integral curve* of X if

$$v_{\gamma,\gamma(\lambda)} = X_{\gamma(\lambda)} \quad \forall\, \lambda \in I.$$

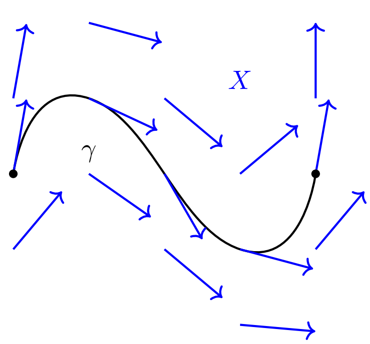


Figure 14: Integral curve γ of the vector field representation of X.

Refer Figure 14, where the tangent vector at every point of the curve γ coincides with the tangent vectors of the vector field X.

Definition 11.4 (Complete vector field)**.** A vector field X is said to be *complete* if all integral curves have $I = \mathbb{R}$.

Theorem 11.5. *A compactly supported vector field is complete.*

Definition 11.6 (Flow of a vector field)**.** The flow of a complete vector field X is a one-parameter family

$$\begin{aligned} h^X \colon \mathbb{R} \times M &\to M \\ (\lambda, p) &\mapsto h^X_\lambda(p) \coloneqq \gamma_p(\lambda), \end{aligned}$$

where $\gamma_p \colon \mathbb{R} \to M$ is the integral curve of X with $\gamma(0) = p$.

From Definition 11.6, we see that for a fixed $\lambda \in \mathbb{R}$, we have $\phi = h^X_\lambda \colon M \to M$, a smooth map from M to itself, and for a set $S \in M$, $h^X_\lambda(S) \neq S$ in general if $X \neq 0, \lambda \neq 0$.

11.4 Lie sublgebras for the Lie algebra $(\Gamma(TM), [\,\cdot\,,\,\cdot\,])$

For a smooth manifold $(M, \mathcal{O}, \mathcal{A})$, we have $\Gamma(TM)$, the set of all smooth vector fields, an $\mathbb{R}$-vector space, as well as a $C^\infty(M)$ module. We also have that for vector fields $X, Y, Z \in \Gamma(X, Y)$, and a smooth function $f \in C^\infty(M)$, $[X, Y]f \coloneqq X(Yf) - Y(Xf)$ is again a smooth vector field, and the Lie bracket $[\,\cdot\,,\,\cdot\,]$ satisfies

1. $[X, Y] = -[Y, X]$,
2. $[\lambda, X + Z, Y] = \lambda[X, Y] + [Z, Y]$, and
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

The tuple $(\Gamma(TM), [\,\cdot\,,\,\cdot\,])$ is therefore a *Lie Algebra*. Now let $X_1, \ldots, X_s$ be s-many vector fields on M, such that $[X_i, X_j] = C^k_{i,j} X_k$, for some structure constants $C^k_{i,j} \in \mathbb{R}$. The span $L = \mathrm{span}_\mathbb{R}\{X_1, \ldots, X_s\}$ with the bracket $[\,\cdot\,,\,\cdot\,]$ together is called a *Lie-subalgebra*.

Example 1. Consider the smooth manifold $(\mathbb{S}^2, \mathcal{O}, \mathcal{A})$, where we have $X_1, X_2, X_3 \in \Gamma\big(T\mathbb{S}^2\big)$, which satisfy

1. $[X_1, X_2] = X_3$,
2. $[X_2, X_3] = X_1$, and
3. $[X_3, X_1] = X_2$.

The Lie-subalgebra $(\mathrm{span}_\mathbb{R}\{X_1, X_2, X_3\}, [\,\cdot\,,\,\cdot\,])$ is called the $so(3)$ Lie-subalgebra. For the chart $(U, x = (\theta, \varphi))$, where $U = x^{-1}((0, \pi) \times (0, 2\pi))$ we indeed have for any $p \in U$,

$$\begin{aligned} X_1(p) &= -\sin(\varphi(p)) \left(\frac{\partial}{\partial \theta}\right)_p - \cot(\theta(p)) \cos(\varphi(p)) \left(\frac{\partial}{\partial \varphi}\right)_p, \\ X_2(p) &= \cos(\varphi(p)) \left(\frac{\partial}{\partial \theta}\right)_p - \cot(\theta(p)) \sin(\varphi(p)) \left(\frac{\partial}{\partial \varphi}\right)_p, \\ X_3(p) &= \left(\frac{\partial}{\partial \varphi}\right)_p. \end{aligned}$$

11.5 Symmetry of a metric

Definition 11.7 (Symmetry of a metric)**.** A finite dimensional Lie-subalgebra $(L, [\,\cdot\,,\,\cdot\,])$, of the Lie algebra $(\Gamma(TM), [\,\cdot\,,\,\cdot\,])$ of a smooth metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$ is said to be a *symmetry* of the metric tensor field g if $\forall\ X \in L$, $\lambda \in \mathbb{R}$, $p \in M$, and $A, B \in T_pM$, we have

$$(\phi^* g)(A, B) \coloneqq g(\phi_*(A), \phi_*(B)) \stackrel{!}{=} g(A, B),$$

where $\phi \coloneqq h^X_\lambda$ as defined in Definition 11.6.

11.6 Lie derivatives

Definition 11.7 does not allow us to trivially check if a metric is symmetric. Defining a Lie derivative will helps us check for symmetries very easily. If we take a vector field $X \in L$ for a Lie-subalgebra $(L, [\,\cdot\,,\,\cdot\,])$ of the Lie algebra $(\Gamma(TM), [\,\cdot\,,\,\cdot\,])$ of a smooth metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$, then the we know that $\forall\ X \in L\ \left(h_\lambda^X\right)^* g - g = 0$, and therefore

$$\mathcal{L}_X g \coloneqq \lim_{\lambda \to 0} \frac{\left(h_\lambda^X\right)^* g - g}{\lambda} = 0, \tag{11.5}$$

if L is a symmetry of g. The quantity $\mathcal{L}_X g$ is called the *Lie derivative* of the metric g with respect to the vector field X.

Definition 11.8 (Lie derivative)**.** The *Lie derivative* on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ sends a pair of a vector field $X \in \Gamma(TM)$, and a (p, q)-tensor field T to a (p, q)-tensor field such that for all (p, q)-tensor fields S, smooth functions $f \in C^\infty(M)$, covectors $\omega_i \in \Gamma(T^*M)\ \forall\ i \in [p]$ and $X_j \in \Gamma(TM)\ \forall\ j \in [q]$,

1. $\mathcal{L}_X f = Xf$,
2. $\mathcal{L}_X Y = [X, Y]$,
3. $\mathcal{L}_X(T, S) = \mathcal{L}_X T + \mathcal{L}_X S$,
4. $$\begin{aligned}\mathcal{L}_X(T(\omega, Y)) = {} & (\mathcal{L}_X T)(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q) \\ & + \sum_{i=1}^{p} T(\ldots, \omega_{i-1}, \mathcal{L}_X \omega_i, \omega_{i+1}, \ldots, X_j, \ldots) + \sum_{j=1}^{q} T(\ldots, \omega_i, \ldots, X_{j-1}, \mathcal{L}_X X_j, X_{j+1}, \ldots),\end{aligned}$$
5. $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$.

Note the difference of the 5^{th} requirement of a Lie derivative in Definition 11.8 with the 4^{th} requirements on the covariant derivative in Definition 7.1. Without the 2^{nd} requirement in Definition 11.8, the restrictions on the Lie derivative is milder than that of a covariant derivative defined in Definition 7.1. With the addition of the requirement that $\mathcal{L}_X Y = [X, Y]$, we do not have any more freedom left in choosing such an operation unlike what we had in the case of the covariant derivative which therefore required the presence of connections to remove the excess freedom. However, unlike the covariant derivative where we could take the covariant derivative with respect to even a tangent vector, the Lie derivative can only be taken with respect to a vector field. The covariant derivative ∇ is $C^\infty(M)$-linear in the first argument, but the Lie derivative $\mathcal{L}$ is not. We observe that for a chart (U, x),

$$\left(\mathcal{L}_X Y\right)^i = X^m \frac{\partial}{\partial x^m} Y^i - Y_s \frac{\partial}{\partial x^s} X^i, \tag{11.6}$$

whereas for the covariant derivative we have

$$\left(\nabla_X Y\right)^i = X^m \frac{\partial}{\partial x^m} Y^i + \Gamma^i_{s,m} X^m Y^s, \tag{11.7}$$

as shown in Equation 7.2. We need the derivative information of the components of X to compute the Lie derivative as it needs to be a vector field. In general, the Lie derivative of a $(1, 1)$-tensor T for example is therefore

$$\left(\mathcal{L}_X T\right)^i_j = X^m \frac{\partial}{\partial x^m} T^i_j - \frac{\partial X^i}{\partial x^s} T^s_j + \frac{\partial X^s}{\partial x^j} T^i_s, \tag{11.8}$$

in contrast to the same for covariant derivatives as in Equation 7.5.
As an application, Equation (11.8) allows us to check/impose conditions under which a metric is symmetric using Equation (11.5).

12 Integration on Manifolds

To define integration of a function on a smooth manifold, we need to add more structures apart from the ones described previously. We need a volume-form, and we need to restrict our atlas further to allow orientation.

12.1 Review of Integration on $\mathbb{R}^d$

For a function $F\colon \mathbb{R} \to \mathbb{R}$, we typically assume a notion of integration, like the Riemann integral or the Lebesgue integral, and the integration of a function over an interval $(a, b) \subseteq \mathbb{R}$ is defined as

$$\int_{(a,b)} F := \int_a^b \mathrm{d}x\ F(x).$$

Similarly, we can extend this notion to $\mathbb{R}^d$ to Cartesian product of intervals $(a_1, b_1) \times \ldots \times (a_d, b_d) \subseteq \mathbb{R}^d$, and define the integral of $F\colon \mathbb{R}^d \to \mathbb{R}$ as

$$\int_{(a_1,b_1)\times\cdots\times(a_d,b_d)} \mathrm{d}^d x\ F(x) := \int_{(a_1,b_1)} \mathrm{d}x^1 \ldots \int_{(a_d,b_d)} \mathrm{d}x^d\ F(X^1, \ldots, x^d).$$

To extend this definition to other types of domains $G \subseteq \mathbb{R}^d$, we introduce an indicator function

$$\begin{aligned} \mu_G\colon \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{else} \end{cases}, \end{aligned}$$

and then define

$$\int_G \mathrm{d}^d x\ F(x) := \int_{-\infty}^{\infty} \mathrm{d}x^1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x^d\ \mu_G(x) \cdot F(x),$$

if it exists.

12.1.1 Change of variables

If we have a function $\phi\colon\ \mathrm{preim}_\phi(G) \to G$, as in Figure 15. the we have the following theorem.

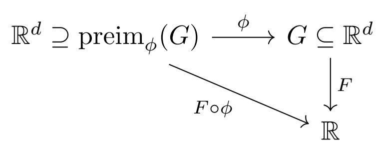


Figure 15: Change of variables in $\mathbb{R}^d$.

Theorem 12.1. *For a set $G \subseteq \mathbb{R}^d$, a function $F\colon G \to \mathbb{R}$, and map $\phi\colon\ \mathrm{preim}_\phi(G) \to G$, we have*

$$\int_G \mathrm{d}^d x\ F(x) = \int_{\mathrm{preim}_\phi(G)} \mathrm{d}^d y\ \left|\det\Big(\left[\partial_b \phi^a\right]_{a,b}(y)\Big)\right| (F \circ \phi)(y),$$

where $\det\Big(\left[\partial_b \phi^a\right]_{a,b}\Big)$ is the Jacobian of the transformation ϕ.

Example 1. For $d = 2$, consider the set $G = \mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$, and the function

$$\begin{aligned} \phi\colon \mathbb{R}_+ \times [(0, \pi) \cup (\pi, 2\pi)] &\to G \\ (r, \varphi) &\mapsto (r\cos\varphi, r\sin\varphi). \end{aligned}$$

Then,

$$\left[(\partial_a \varphi^b)(r, \varphi)\right]_{a,b} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{bmatrix}, \quad \text{and}$$

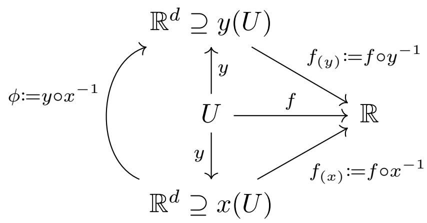
$$\left|\det\left(\left[(\partial_a\varphi^b)(r,\varphi)\right]_{a,b}\right)\right| = r.$$

Therefore,

$$\int_G \mathrm{d}x^1\,\mathrm{d}x^2\ F(x^1,x^2) = \int_{r=0}^{\infty}\int_{\varphi=0}^{2\pi} \mathrm{d}r\,\mathrm{d}\varphi\ r\cdot F(r\cos\varphi, r\sin\varphi).$$

12.2 Need for some extra structure to integrate

For a smooth manifold $(M,\mathcal{O},\mathcal{A})$, a function $f\colon M\to\mathbb{R}$, and charts $(U,x),(U,y)\in\mathcal{A}$.



Observe that

$$\begin{aligned}
\int_{y(U)} \mathrm{d}^d\beta\ f_{(y)}(\beta) &= \int_{x(U)} \mathrm{d}^d\alpha\ \left|\det\left(\left[\partial_a\left(y^b\circ x^{-1}\right)(\alpha)\right]\right)\right|\left(f_{(y)}\circ\left(y\circ x^{-1}\right)\right)(\alpha)\\
&= \int_{x(U)} \mathrm{d}^d\alpha\ \left|\det\left(\frac{\partial y^b}{\partial x^a}\right)_{x^{-1}(\alpha)}\right|\left(f\circ x^{-1}\right)(\alpha)\\
&\neq \int_{x(U)} \mathrm{d}^d\alpha\ f_{(x)}(\alpha),
\end{aligned}$$

therefore an attempt to define the integral $\int_U f$ as an integral over some chart $\int_{x(U)}\mathrm{d}^d\alpha\ f_{(x)}(\alpha)$, fails. Therefore, if somehow we are able to introduce an extra term to cancel out the determinant term, we will have a consistent definition of an integral over a manifold. However, it turns out that there is no such object like this and the smooth manifold structure alone is not sufficient to introduce the notion of integration over a manifold.

12.3 Volume forms

To introduce the notion of integration on a manifold, we need to define what is called a volume form.

Definition 12.2. On a smooth manifold $(M,\mathcal{O},\mathcal{A})$, we call a $(0,\dim M)$-tensor field Ω, a *volume form* if

1. Ω vanishes nowhere, and
2. Ω is totally anti-symmetric, i.e.,

$$\Omega\left(\ldots,\underbrace{X}_{i^{\text{th}}\text{ position}},\ldots,\underbrace{Y}_{j^{\text{th}}\text{ position}},\ldots\right) = -\Omega\left(\ldots,\underbrace{Y}_{i^{\text{th}}\text{ position}},\ldots,\underbrace{X}_{j^{\text{th}}\text{ position}},\ldots\right),$$

   for all distinct indices $i\neq j\in[\dim M]$.

From Definition 12.2, we see that the chart induced components of Ω, satisfy $\Omega_{i_1,\ldots,i_d} = \Omega_{[i_1,\ldots,i_d]}$. We already have defined a metric structure in §10, so the question remains if we can construct a volume form Ω out of a metric g on the smooth manifold $(M,\mathcal{O},\mathcal{A},g)$. To define a volume form out of a metric, we need Levi-Civita symbols ϵ with $\dim M$ many indices, which is defined as $\epsilon_{1,2,\ldots,d}=1$, and satisfies $\epsilon_{i_1,\ldots,i_d} = \epsilon_{[i_1,\ldots,i_d]}$. Then, the coordinates of the volume form Ω under a chart (U,x) can be defined as

$$\Omega_{(x),i_1,\ldots,i_d} := \sqrt{\left|\det\left(\left[g_{(x),i,j}\right]_{i,j}\right)\right|}\,\epsilon_{i_1,\ldots,i_d}. \tag{12.1}$$

Under a change of charts to (U, y), the components of Ω on this chart would be

$$\begin{aligned}
\Omega_{(y),i_1,\dots,i_d} &= \sqrt{\left|\det\Big(\big[g_{(y),i,j}\big]_{i,j}\Big)\right|}\,\epsilon_{i_1,\dots,i_d} \\
&= \sqrt{\left|\det\left(\left[g_{(x),m,n}\left(\frac{\partial x^m}{\partial y^i}\right)\left(\frac{\partial x^n}{\partial y^j}\right)\right]_{i,j}\right)\right|}\left(\frac{\partial y^{m_1}}{\partial x^{i_1}}\right)\cdots\left(\frac{\partial y^{m_d}}{\partial x^{i_d}}\right)\epsilon_{m_1,\dots,m_d} \\
&= \sqrt{\left|\det\Big(\big[g_{(x),i,j}\big]_{i,j}\Big)\right|}\sqrt{\det\left(\left[\left(\frac{\partial x^a}{\partial y^b}\right)\right]\right)^2}\det\left(\left[\left(\frac{\partial y^a}{\partial x^b}\right)\right]_{a,b}\right)\epsilon_{i_1,\dots,i_d} \\
&= \sqrt{\left|\det\Big(\big[g_{(x),i,j}\big]_{i,j}\Big)\right|}\,\epsilon_{i_1,\dots,i_d}\cdot\operatorname{sign}\left(\det\left(\left[\left(\frac{\partial y^a}{\partial x^b}\right)\right]_{a,b}\right)\right).
\end{aligned}$$

Therefore, the volume form as defined in Equation (12.1) is well defined if $\det\left(\left[\left(\frac{\partial y^a}{\partial x^b}\right)\right]_{a,b}\right) > 0$ for any pair of charts (U, x) and (U, y). We can therefore restrict the smooth atlas $\mathcal{A}$ to a sub-atlas $\mathcal{A}^{\uparrow} \subseteq \mathcal{A}$, such that for any two charts $(U, x), (V, y) \in \mathcal{A}^{\uparrow}$, we have $\det\left(\left[\left(\frac{\partial y^a}{\partial x^b}\right)\right]_{a,b}\right) > 0$. Such an atlas $\mathcal{A}^{\uparrow}$ is called an *oriented atlas*. Therefore on a smooth oriented metric manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$ we do have a well-defined volume form.

Remark 12.3. Note that we take the determinant of the matrix constructed by the components of the metric in a chart, and not the determinant of the bilinear map itself. We can do this as long as we are able to obtain a chart consistent definition.

Definition 12.4 (Scalar density)**.** Let Ω be a volume form on an oriented manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$, and let (U, x) be a chart. Then, define

$$\omega_{(x)} \coloneqq \Omega_{i_1,\dots,i_d}\epsilon^{i_1,\dots,i_d},$$

where again $\epsilon^{1,2,\dots,d} = 1$, and $\epsilon^{i_1,\dots,i_d} = \epsilon^{[i_1,\dots,i_d]}$. Then one can show that $\omega_{(y)} = \det\left(\left[\left(\frac{\partial x^a}{\partial y^b}\right)\right]_{a,b}\right)\omega_{(x)}$, and is called the *scalar density*.

12.4 Integration on a chart domain

Definition 12.5. Let Ω be a volume form on an oriented manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$, and let (U, x) be a chart. Then we define

$$\int_U f \coloneqq \int_{x(U)} \mathrm{d}^d\alpha\ \omega_{(x)}\big(x^{-1}(\alpha)\big) f_{(x)}(\alpha), \tag{12.2}$$

where $f_{(x)} = f \circ x^{-1}$.

The integration over a chart as defined in Definition 12.5 is well defined since for two charts $(U, x), (U, y) \in \mathcal{A}^{\uparrow}$,

$$\begin{aligned}
\int_U f &= \int_{y(U)} \mathrm{d}^d\beta\ \omega_{(y)}\big(y^{-1}(\beta)\big) f_{(y)}(\beta) \\
&= \int_{x(U)} \mathrm{d}^d\alpha \left|\det\left(\frac{\partial y}{\partial x}\right)_{x^{-1}(\alpha)}\right| \det\left(\frac{\partial x}{\partial y}\right)_{x^{-1}(\alpha)} \omega_{(x)}\big(x^{-1}(\alpha)\big) f_{(x)}(\alpha) \\
&= \int_{x(U)} \mathrm{d}^d\alpha\ \omega_{(x)}\big(x^{-1}(\alpha)\big) f_{(x)}(\alpha),
\end{aligned}$$

is now chart independent.
On an oriented metric manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$, we have

$$\int_U f \coloneqq \int_{x(U)} \mathrm{d}^d\alpha\ \sqrt{\left|\det\Big(\big[g_{(x),i,j}\big]_{i,j}\big(x^{-1}(\alpha)\big)\Big)\right|} f_{(x)}(\alpha).$$

As a short-hand, we write the above equation as $\int_U f = \int_{x(U)} \mathrm{d}^d\alpha\ \sqrt{g} f_{(x)}(\alpha)$.

12.5 Integration on the entire manifold

We can integrate over charts, but it is not immediate to see how this alone can be help us to integrate over the entire manifold since at non-empty chart intersections, we would add up the same domain multiple times. We cannot cut out the common intersection regions since intersections may not be open sets. Therefore, we require that the manifold admits a *partition of unity.*
For an oriented manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$, we need to require that for any finite sub-atlas $\mathcal{A}' \subseteq \mathcal{A}^{\uparrow}$, there exists continuous functions $\rho_i \colon U_i \to \mathbb{R}$ satisfies

$$\sum_{p \in U_i} \rho_i(p) = 1 \quad \forall\, p \in M,$$

where $\mathcal{A}' = \{(U_i, x_i)\}_i$. Then $\{\rho_i\}_i$ is a partition of unity.

Definition 12.6. The integration of a function f over an oriented manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$ with a partition of unity $\{\rho_i\}_i$ is defined as

$$\int_M f \coloneqq \sum_{\rho_i \in \{\rho_j\}_j} \int_{U_i} (\rho_i \cdot f).$$

Remark 12.7. The finiteness of $\mathcal{A}'$ itself is not necessary, since what we only need is that at every point in the manifold there exists finitely many charts that contain it, so that the summation does not diverge. The existence of finite charts for every point is a weaker requirement, and this notion is called paracompactness.

13 Relativistic Spacetime

In § 9, we saw that the Newtonian Spacetime is the tuple $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$, where $(M, \mathcal{O}, \mathcal{A})$ is a smooth 4-manifold, ∇ is a torsion-free connection, and $t \in C^\infty(M)$ satisfying $\mathrm{d}t \neq 0$, and $\nabla \mathrm{d}t = 0$.

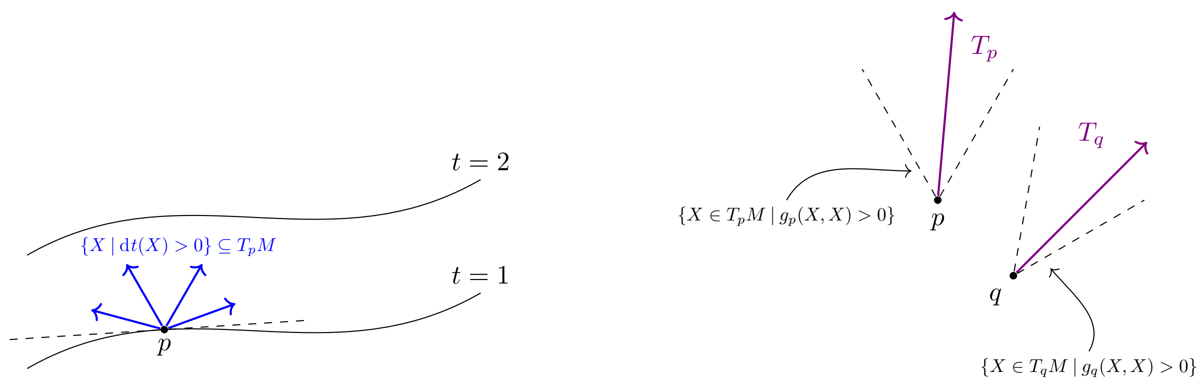
Relativistic spacetime however is a 6-tuple $(M, \mathcal{O}, \mathcal{A}, \nabla, g, T)$, where like Newtonian spacetime, $(M, \mathcal{O}, \mathcal{A})$ is a smooth 4-manifold, ∇ is a torsion-free connection. In addition, it has a Lorentzian metric g (with signature $(+, -, -, -)$), and a time orientation $T \in TM$ which is a smooth vector field. It is to observe that the role played by t as a time function in Newtonian spacetime is now played by the interplay of two different structures, a Lorentzian metric g and a time orientation.

Fact 13.1. Not all smooth manifolds can carry a Lorentzian metric.

13.1 Time orientation

Definition 13.2. Let $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$ be an oriented Lorentzian smooth manifold, then a time orientation is given by a vector field T that

1. does not vanish anywhere, and
2. $g(T, T) > 0$.



(a) The half-space of future directed tangent vectors at point p, in Newtonian spacetime.

(b) Half-cones induced by g and T in tangent spaces of points p and q, in relativistic spacetime.

Figure 16: Difference in Newtonian and Relativistic spacetime.

As demonstrated in Figure 16, the combination of g and T gives a distribution of half-cones in the tangent spaces of each point in relativistic spacetime. Whereas, the time function t gives a distribution of half-spaces in the tangent space of each point in Newtonian spacetime.

Note that without the time orientation vector field T, the set $\{X \in T_pM \mid g_p(X, X) > 0\}$ at a point $p \in M$ is an open half-cone in the tangent space T_pM at p. In order for particles to 'run forward' in time, we need to select one half of the cone, and this is done using T.

This definition of spacetime has been made to enable the following physical postulates:

P1. The worldline $\gamma \colon I \to M$ of a 'massive' particle satisfies for any $\lambda \in I$,

   (a) $g_{\gamma(\lambda)}\big(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\big) > 0$, and
   (b) $g_{\gamma(\lambda)}\big(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\big) > 0$,

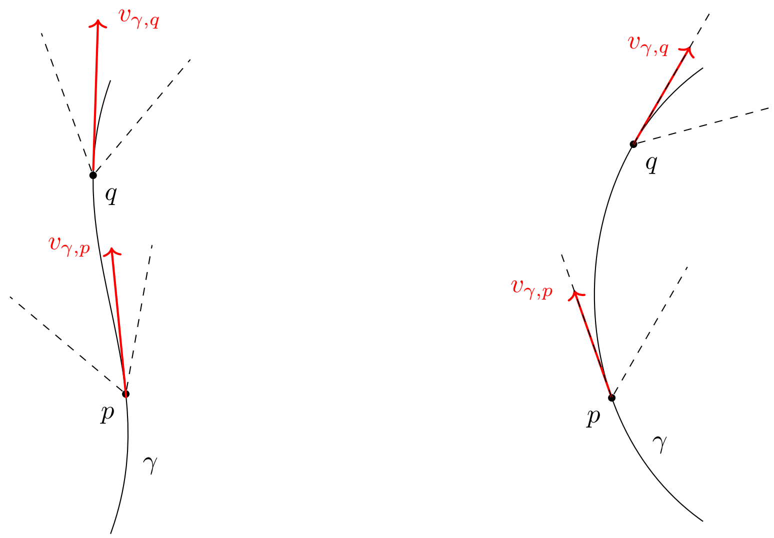
   as shown in Figure 17a.

P2. The worldline $\gamma \colon I \to M$ of 'massless' particles satisfies for any $\lambda \in I$,

   (a) $g_{\gamma(\lambda)}\big(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\big) = 0$, and

(b) $g_{\gamma(\lambda)}\big(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\big) > 0$.

as shown in Figure 17b. Note that we could not talk about massless particles in Newtonian spacetime, but in relativistic spacetime we can.



(a) Worldlines of 'massive' particles.

(b) Worldlines of 'massless' particles.

Figure 17: Depiction of worldlines in relativistic spacetime.

13.2 Observers

Consider a 4-dimensional relativistic spacetime manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$.

Definition 13.3 (Observer)**.** An *observer* is a smooth curve $\gamma\colon I \to M$ with $g(v_\gamma, v_\gamma) = 1$ and $g(T, v_\gamma) > 0$, together with a smoothly varying frame, i.e., a choice of basis $v_{\gamma,\gamma(\lambda)} \equiv e_0(\lambda)$, $e_1(\lambda)$, $e_2(\lambda)$ and $e_3(\lambda)$, at each $T_{\gamma(\lambda)}M$ where the observer's worldline passes, if $g(e_a(\lambda), e_b(\lambda)) = \eta_{a,b}$, where $\big[\eta_{a,b}\big]_{a,b} = \mathrm{diag}\big([1, -1, -1, -1]\big)$.

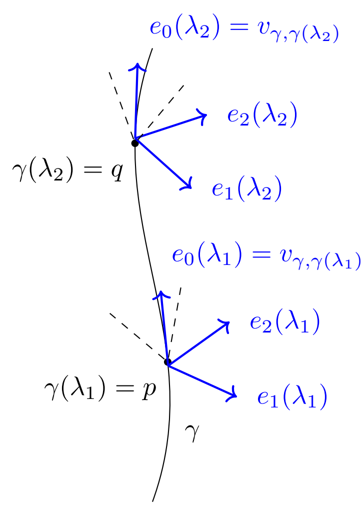


Figure 18: Observer in Relativistic spacetime

The definition of an observer has been made to enable the following physical postulates:

P3. A 'clock' carried by a specific observer (γ, e) will measure a 'proper time'

$$\tau := \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda \sqrt{g_{\gamma(\lambda)}\big(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\big)}$$

between two 'events' $\gamma(\lambda_0)$ and $\gamma(\lambda_1)$ corresponding to starting and stopping the clock.

Example 1. Consider $M = \mathbb{R}^4$, $\mathcal{O} = \mathcal{O}_{\text{standard}}$, with a chart $(U = \mathbb{R}^4, x = \mathrm{id}_{\mathbb{R}^4}) \in \mathcal{A}^{\uparrow}$ such that $g_{(x),i,j} = \eta_{i,j}$ as defined in Definition 13.3, and $T^i_{(x)} = (1,0,0,0)^i$. The Christoffel symbols therefore will satisfy $\Gamma^i_{(x),j,k} = 0$ everywhere, and therefore he Riemann curvature Riem $= 0$. This spacetime is 'flat', and is the domain of Special Relativity. Consider, two observers $\gamma\colon (0,1) \to M$, where its components

$$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i,$$

and $\delta\colon (0,1) \to M$, where its components satisfy for some $\alpha \in (0,1)$

$$\delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \text{if } \lambda \le \frac{1}{2}, \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \text{if } \lambda > \frac{1}{2}. \end{cases}$$

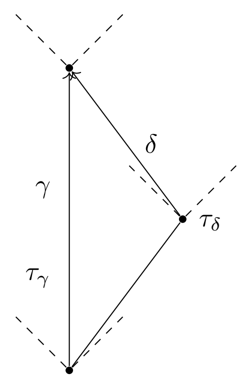


Figure 19: The twin paradox example

Then, the times takes by the two observers will be

$$\begin{aligned} \tau_\gamma &:= \int_0^1 \mathrm{d}\lambda \sqrt{g_{(x),i,j}\dot\gamma^i_{(x)}\dot\gamma^j_{(x)}} \\ &= \int_0^1 \mathrm{d}\lambda\ 1 = 1, \quad \text{and} \\ \tau_\delta &:= \int_0^{\frac{1}{2}} \mathrm{d}\lambda \sqrt{g_{(x),i,j}\dot\delta^i_{(x)}\dot\delta^j_{(x)}} + \int_{\frac{1}{2}}^1 \mathrm{d}\lambda \sqrt{g_{(x),i,j}\dot\delta^i_{(x)}\dot\delta^j_{(x)}} \\ &= \int_0^{\frac{1}{2}} \mathrm{d}\lambda \sqrt{1-\alpha^2} + \int_{\frac{1}{2}}^1 \mathrm{d}\lambda \sqrt{1-(-\alpha)^2} \\ &= \int_0^1 \mathrm{d}\lambda \sqrt{1-\alpha^2} = \sqrt{1-\alpha^2}. \end{aligned}$$

respectively. Observe that the time measured by the two observers are different where they meet back at the same point in spacetime. The closer α gets to 1, the smaller is the time measured by the second observer as its tangent vector approaches the boundary of the cone at every points along its trajectory in spacetime.

Note that 'time' is a derived notion in the context of relativistic spacetime, where it is measured by a 'clock'. Time depends on the worldline that the clock takes in relativistic spacetime, whereas in Newtonian spacetime, every point $p \in M$ is associated with a time $t(p)$ and the time between two points is the difference in the time functions at the two points which is independent of the path taken if by an observer.

P4. Let (γ, e) be an observer and let δ be a 'massive' particle worldline, that is parameterized such that $g(v_\delta, v_\delta) = 1$. This normalization is done to parameterize the worldline according to the time the clock would show that travels with the particle. Suppose the observer and the particle meet somewhere in spacetime such that $\delta(\tau_2) = p = \gamma(\tau_1)$. Then v_δ is the spacetime 'velocity' to the worldline of the massive particle.

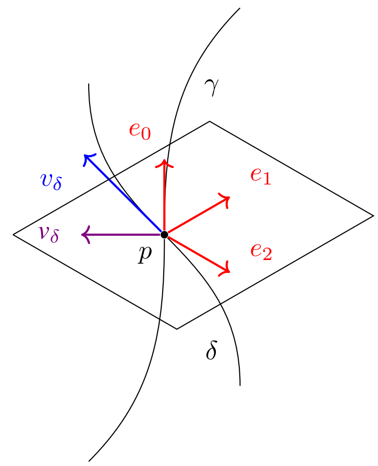


Figure 20: Spacetime velocity v_δ and spatial velocity v_δ

The observer (γ, e) measures the spatial velocity of the massive particle as

$$\mathit{v}_{\delta,\delta(\tau_2)} := \epsilon^{\alpha}\big(v_{\delta,\delta(\tau_2)}\big)e_\alpha,$$

where α runs in [3], and $\{\epsilon^a\}_{a=0}^3$ is the unique dual basis of $\{e_a\}_{a=0}^3$ of T_pM.

Note that the spacetime velocity is objective as it is the tangent vector of the worldline which is an objective reality. The spatial velocity is constructed with respect to an observer, and different observers can construct different spatial velocities from the same spacetime velocity.

As a consequence, an observer (γ, e) will extract quantities measurable in their laboratory from objective spacetime quantities using the above postulates.

Example 2. Consider the objective Faraday $(0, 2)$-tensor of electromagnetism

$$\big[F(e_a, e_b)\big]_{a,b} = \big[F_{a,b}\big]_{a,b} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_2 & B_2 & -B_3 & 0 \end{bmatrix}.$$

The coordinates of the tensor are dependent on the observer as they depend on $\{e_a\}_{a=0}^3$, the observer's frame, and therefore different observers can measure the same tensor differently.

13.3 Role of the Lorentz Transformations

Let (γ, e) and $(\tilde{\gamma}, \tilde{e})$ be observers with $\gamma(0) = \tilde{\gamma}(0)$.
Therefore, both $\{e_a\}_{a=0}^3$, and $\{\tilde{e}_a\}_{a=0}^3$ are both bases for the same tangent space $T_{\gamma(0)}M$, and we can use a transformation to change basis and write one in terms of the other, i.e., for some $\Lambda \in \mathrm{GL}(4)$,

$$\tilde{e}_a = \Lambda_a^b e_b.$$

Now note that

$$\begin{aligned} \eta_{a,b} &= g(\tilde{e}_a, \tilde{e}_b) \\ &= g(\Lambda_a^m e_m, \Lambda_b^n e_n) \end{aligned}$$

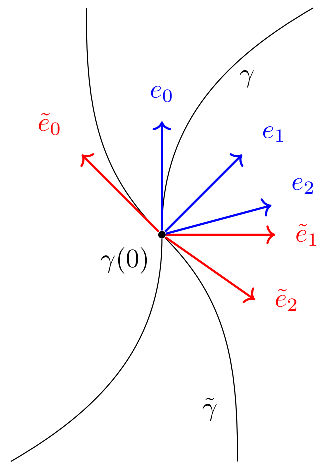


Figure 21: Two different observer frames at the same point

$$= \Lambda_a^m \Lambda_b^n g(e_m, e_n)$$
$$= \Lambda_a^m \Lambda_b^n \eta_{m,n}.$$

Lorentz transformations relate the frames of any two observers at the same point. It has nothing to do with frames at different points and must not be used across points in spacetime.

References

[1] The WE-Heraeus International Winter School on Gravity and Light. Central lecture course. https://www.youtube.com/playlist?list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_, 2015.